# Orthogonal matrices

## 1. Orthogonal matrices in general

*Definition.* An $n \times n$ real matrix $P$ is *orthogonal* if its rows are an orthonormal set of vectors: mutually perpendicular and of length 1. A homogeneous linear transformation is said to be orthogonal if its matrix is.

Examples are the identity matrix $I$ and rotations and reflections (discussed below), for any $n$.

*Theorem.* 1.1 . For an $n \times n$ real matrix $P$ and for $T$ given by $T(\mathbf{x}) = \mathbf{x}P$, the following are equivalent:

(1) $P$ is orthogonal, i.e., the rows of $P$ are orthonormal;

(2) $P^t$ is orthogonal, i.e., the columns of $P$ are orthonormal;

(3) $P^{-1} = P^t$ (so $P^{-1}$ exists and is trivial to find);

(4) $PP^t = I$;

(5) $P^tP = I$;

(6) $T$ preserves dot products, i.e., $T(\mathbf{x}) \cdot T(\mathbf{y}) = \mathbf{x} \cdot \mathbf{y}$;

(7) $T$ preserves lengths, i.e., $|T(\mathbf{x})| = |\mathbf{x}|$ for all $\mathbf{x}$;

(8) $T$ preserves distances, i.e., $|T(\mathbf{x}) - T(\mathbf{y})| = |\mathbf{x} - \mathbf{y}|$, so that $T$ is a *rigid motion* or *isometry* or *congruence*.

*Corollary* 1.2 . If $P$ is orthogonal, then $\det P = \pm 1$.

*Corollary* 1.3 . (a) The product of two orthogonal matrices is orthogonal. (b) The inverse of an orthogonal matrices is orthogonal.

*Remark.* The standard name “orthogonal matrix” is unfortunate; “orthonormal matrix” would have been better.

## 2. Rotation matrices

*Definition* 2.1 . A *rotation matrix* is an orthogonal matrix of determinant $+1$, rather than $-1$. (Note: The identity matrix *is* considered a rotation matrix.)

*Proposition* 2.2 . (a) The product of two $n \times n$ rotation matrices is a rotation matrix. (b) The inverse of a rotation matrix is a rotation matrix. (In fact, it's the transpose.)

2.3 **Rotation matrices in $\mathbf{R}^2$**

Our notation will be $R_\theta = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$, giving a rotation about the origin through an angle $\theta$, measured counterclockwise.

*Proposition* 2.4 . All $2 \times 2$ rotation matrices are of this form.

2.5 **Rotation matrices in $\mathbf{R}^3$**

By $R_\theta^{x\to y}$ we will mean the rotation matrix that gives a rotation about the $z$-axis by an angle $\theta$ measured starting from the $x$-axis and rotating towards the $y$-axis. Its effect on the $x, y$-plane is the same as the effect of $R_\theta$. Because $(1, 0, 0)$ and $(0, 1, 0)$ are moved in the $x, y$-plane as if $R_\theta$ were being applied, and because $(0, 0, 1)$ is left fixed, $R_\theta^{x\to y} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

$R_\theta^{y\to x}$ is the inverse of $R_\theta^{x\to y}$. Thus $R_\theta^{y\to x}$ gives the rotation about the $z$-axis by an angle $\theta$ measured starting from the $y$-axis and rotating towards the $x$-axis.

$R_\theta^{x\to z}$, $R_\theta^{z\to x}$, $R_\theta^{y\to z}$, and $R_\theta^{z\to y}$ are defined similarly.

2.6 *Remarks.*

(1) *Not all* $3\times 3$ rotation matrices are of these forms, but all can be obtained as products of matrices of these forms.

(2) The word *axis* can mean two things: (i) a coordinate axis, i.e., the $x$-axis, $y$-axis, or $z$-axis; (ii) the axis of a rotation in $\mathbf{R}^3$, i.e., a line of points that are left fixed (are not moved) by the rotation. It is a fact that for every $3 \times 3$ matrix the corresponding rotation in $\mathbf{R}^3$ has an axis. (See Problem F-22.)

(3) Because a rotation matrix has positive determinant, the rotation it describes preserves not only angles (like all orthogonal transformations) but also orientation, and so it preserves cross products. Because the rows of a rotation matrix are the images of $\mathbf{i}, \mathbf{j}, \mathbf{k}$, they have the same cross-product relations as $\mathbf{i}, \mathbf{j}, \mathbf{k}$. In particular, since $\mathbf{i} \times \mathbf{j} = \mathbf{k}$, the third row of a rotation matrix is the cross product of the first and second rows. (What about columns?)

## 3. Rotations as transformations

The transformation $T(\mathbf{x}) = \mathbf{x}P$ that comes from a rotation matrix $P$ is a rotation of a special kind: Because it is a homogeneous linear transformation, it must leave the origin fixed. Thus in $\mathbf{R}^2$, a rotation matrix gives a rotation about the origin (a rotation whose center is the origin); in $\mathbf{R}^3$, a rotation matrix gives a rotation whose axis goes through the origin.

In graphics, however, it is often necessary to rotate in $\mathbf{R}^2$ about other centers and in $\mathbf{R}^3$ about axes not through the origin. Precisely how to make such rotations will be discussed as part of the study of "affine transformations". For now, just consider this definition:

*Definition* 3.1 . A transformation $\mathbf{R}^n \to \mathbf{R}^n$ is a *rotation* if it is rigid, preserves orientation, and leaves at least one point fixed (i.e., takes the point to itself).

If a rotation leaves the origin fixed, then it is a homogeneous linear transformation coming from a rotation matrix. (This fact is not hard to prove.)

**Agreement**. In talking about rotations, let's assume we *are* talking about rotations that leave the origin fixed, unless it is obvious that we are not.

## 4. Reflections

In $\mathbf{R}^2$, a homogeneous linear transformation $T$ is a *reflection* if there is some line $L$ through the origin such that for each $\mathbf{x}$, $T(\mathbf{x})$ is the reflection of $\mathbf{x}$ with $L$ as a mirror. In other words, the line segment from $\mathbf{x}$ to $T(\mathbf{x})$ has perpendicular bisector $L$.

In $\mathbf{R}^3$, a homogeneous linear transformation is a reflection if there is a plane $H$ through the origin such that for each $\mathbf{x}$, $T(\mathbf{x})$ is the reflection of $\mathbf{x}$ with $H$ as a mirror. In other words, the line segment from $\mathbf{x}$ to $T(\mathbf{x})$ is perpendicular to $H$ and is bisected by $H$.

It is a fact that in $\mathbf{R}^2$ any orthogonal matrix is either a rotation or a reflection, but as you will see from an exercise, the situation in $\mathbf{R}^3$ is not so simple.

Suppose the homogeneous linear transformation $T$ is a reflection. Let $P$ be the matrix of $T$, and let $\mathbf{N}$ be any normal to the mirror, $\mathbf{n}$ a unit normal. Some facts:

1. The matrix $P$ of $T$ is an orthogonal matrix (because reflections are rigid).
2. $\det P = -1$ (because reflections reverse orientation).
3. $P^2 = I$ (because doing a reflection twice has no effect).
4. $T$ leaves fixed each point of its mirror.
5. $T(\mathbf{N}) = -\mathbf{N}$, $T(\mathbf{n}) = -\mathbf{n}$
6. $T$ has the equation $T(\mathbf{x}) = \mathbf{x}M$ for $M = I - 2\mathbf{n}^t\mathbf{n} = I - \frac{2}{\mathbf{N}\cdot\mathbf{N}}\mathbf{N}^t\mathbf{N}$.

Another way to say (4) is that each vector in the mirror is an eigenvector of $T$ for the eigenvalue 1. Another way to say (5) is that each vector perpendicular to the mirror is an eigenvector for the eigenvalue $-1$.

The fact (6) is easy to apply. For example, if the mirror plane is $x+2y+2z = 0$, then $\mathbf{N} = (1,2,2)$ and the matrix is $I - \frac{2}{\mathbf{N}\cdot\mathbf{N}}\mathbf{N}^t\mathbf{N} = I - \frac{2}{9}\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}\begin{bmatrix} 1 & 2 & 2 \end{bmatrix}$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \frac{2}{9}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 4 & 4 \\ 2 & 4 & 4 \end{bmatrix} = \begin{bmatrix} \frac{7}{9} & -\frac{4}{9} & -\frac{4}{9} \\ -\frac{4}{9} & \frac{1}{9} & -\frac{8}{9} \\ -\frac{4}{9} & -\frac{8}{9} & \frac{1}{9} \end{bmatrix}.$$

This method also works in $\mathbf{R}^2$, where the mirror is a line with normal $\mathbf{N}$. It is usually easier than the three-step method of "rotate, easy reflection, rotate back".

The matrices of reflections are also useful in numerical analysis, where they are called *Householder transformations.*

Just as for rotations, it is possible to talk about reflections that do not leave the origin fixed, i.e., whose mirrors do not go through the origin. In this case, they are not homogeneous linear transformations and cannot be described by orthogonal matrices. Also as for rotations, let's agree that we *are* talking about the homogeneous linear case unless it's obvious that we are not.

## 5. Problems

**Problem F-1.** Prove Theorem 1.1. Suggested: (1) $\Leftrightarrow$ (4) $\Leftrightarrow$ (3) $\Leftrightarrow$ (5) $\Leftrightarrow$ (2) and (4) $\Rightarrow$ (6) $\Rightarrow$ (7) $\Rightarrow$ (8) $\Rightarrow$ (1).

**Problem F-2.** Prove Corollaries 1.2 and 1.3.

**Problem F-3.** Prove Proposition 2.2.

**Problem F-4.** Prove Proposition 2.4.

**Problem F-5.** A *permutation matrix* is a matrix $P$ whose entries are 0 except for a single 1 in each row and column, for example, $P = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$.

(a) For this $P$, describe the action of the corresponding transformation $T$ when applied to a vector $(a, b, c, d)$.

(b) Find the entries of $P^{-1}$, row by row. Do this aloud, without writing anything.

**Problem F-6.** The following are some interesting cases. Verify directly that each of them is orthogonal. They are really all rotations about an axis through the origin and $(1, 1, 1)$.

(a) A rotation by 30° is $\frac{1}{3}\begin{bmatrix} (1+\sqrt{3}) & 1 & (1-\sqrt{3}) \\ (1-\sqrt{3}) & (1+\sqrt{3}) & 1 \\ 1 & (1-\sqrt{3}) & (1+\sqrt{3}) \end{bmatrix}$

(b) A rotation by 60° is $\frac{1}{3}\begin{bmatrix} 2 & 2 & -1 \\ -1 & 2 & 2 \\ 2 & -1 & 2 \end{bmatrix}$

(c) A rotation by 90° is $\frac{1}{3}\begin{bmatrix} 1 & (1+\sqrt{3}) & (1-\sqrt{3}) \\ (1-\sqrt{3}) & 1 & (1+\sqrt{3}) \\ (1+\sqrt{3}) & (1-\sqrt{3}) & 1 \end{bmatrix}$

(d) A rotation by 120° is $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ (In this example, could you find the matrix by direct reasoning? What happens to the $x$-, $y$-, and $z$-axes when you rotate by 120° about an axis along $(1, 1, 1)$?)

**Problem F-7.** Suppose $A$ is defined to be the product $A =$

$$\begin{bmatrix} .6 & .8 & 0 \\ -.8 & .6 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} .6 & 0 & .8 \\ 0 & 1 & 0 \\ -.8 & 0 & .6 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & .6 & .8 \\ 0 & -.8 & .6 \end{bmatrix} = \begin{bmatrix} 0.36 & 0.096 & 0.928 \\ -0.48 & 0.872 & 0.096 \\ -0.8 & -0.48 & 0.36 \end{bmatrix}.$$

(a) Explain whether $A$ is orthogonal, and why.

(b) Find the inverse of $A$ explicitly.

(c) Is $A$ a rotation? Explain why or why not.

**Problem F-8.** (a) Write down these matrices, with explicit entries: $R_\theta^{x \to y}$, $R_\theta^{y \to z}$, $R_\theta^{x \to z}$. To get the signs of the entries, imagine an example where $0 < \theta < \frac{\pi}{2}$ and decide whether the images of $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ have any negative coordinates.

(b) For each, say whether the rotation is clockwise or counterclockwise as seen looking towards the origin from a point on the positive half of its fixed axis. (For example, for $R_\theta^{x \to y}$, the point would be $(0, 0, c)$ with $c > 0$.) Any surprises?

**Problem F-9.** For a given $\theta$, you can write six matrix expressions such as $R_\theta^{x \to y}$, $R_\theta^{y \to x}$, $R_\theta^{x \to z}$, etc. Their inverses give six more matrix expressions. Another twelve expressions are formed by putting $-\theta$ for $\theta$. Break these twenty-four matrix expressions into separate lists so that the expressions in each list are necessarily equal but any two matrices from different lists are not equal in general. For example, $R_\theta^{y \to z}$ and $(R_\theta^{z \to y})^{-1}$ are equal and so are in the same list.

**Problem F-10.** (a) In $\mathbf{R}^2$, clearly $R_{\theta+\phi} = R_\theta R_\phi$. By writing out these matrices and performing the matrix multiplication, derive the laws for the sine and cosine of the sum of two angles.

(b) Similarly, use $R_{\pi-\theta}$ to find formulas for $\cos(\pi - \theta)$ and $\sin(\pi - \theta)$.

(c) If you need the formula for $\cos(\theta + \frac{\pi}{2})$ and don't remember it, what is a simple way to find it?

**Problem F-11.** (a) Find all $2 \times 2$ rotation matrices that are also diagonal. (b) Find all $3 \times 3$ rotation matrices that are diagonal. (c) What about the $n \times n$ case?

**Problem F-12.** To find the inverse of a $2 \times 2$ rotation matrix or a standard $3 \times 3$ rotation matrix (such as $R_\theta^{y \to z}$), it seems to work just to reverse the sign of all off-diagonal entries. Find an example to show that this is not a valid method for all rotations, i.e., find an example of a rotation matrix for which reversing the sign of off-diagonal entries is not the same as taking the transpose.

**Problem F-13.** Show that if the rows of a square matrix are mutually orthogonal and have the *same length*, then the same is true about the columns. (Method: Start by scaling the entries of the matrix uniformly to get a matrix of a kind you know about.)

**Problem F-14.** (a) In $\mathbf{R}^2$, if the list of vertices of a square starts with $(0,0)$ and $(a,b)$, going counterclockwise, what are the remaining two vertices? (Suggestion: The vertex opposite $(a,b)$ can be obtained by rotating $(a,b)$ by $90°$ about the origin.)

(b) In $\mathbf{R}^2$, the *standard unit square* is the square with vertices $(0,0)$, $(1,0)$, $(1,1)$, and $(0,1)$. Find a matrix that takes the standard unit square to the square mentioned in (a).

**Problem F-15.** Matrices of the form $\begin{bmatrix} a & b \\ -b & a \end{bmatrix}$ are useful in graphics.

(a) Suppose numbers $a$ and $b$ are given, not both zero. Find the entries of a rotation matrix that takes $(1,0)$ to a unit vector in the direction of $(a,b)$. (You do not need to express the angle of the rotation.)

(b) Show that a matrix $A$ of the form mentioned is equal to a rotation matrix times a scalar matrix $rI$ with $r > 0$. (Thus $\mathbf{x} \to \mathbf{x}A$ preserves shapes and orientation while expanding or contracting the size uniformly.)

**Problem F-16.** Matrices of the form $\begin{bmatrix} a & b \\ -b & a \end{bmatrix}$, mentioned in Problem F-15, are also useful mathematically. (a) Show that if two matrices of this form are added, you get another matrix of this form. (b) Show that if two matrices of this form are multiplied, you get another matrix of this form. (c) Show that the inverse of a matrix of this form is again of this form. (d) Two matrices $A, B$ are said to commute if $AB = BA$. Show that any two matrices of this form commute. (e) Show that any $2 \times 2$ rotation matrix has this form. (f) Show that the matrices of this form are precisely the matrices that commute with $R_{90°}$. (*Note.* Let's take $a, b$ to be real numbers. By (a)-(d), you can do "arithmetic" with these matrices. It turns out that they work exactly like complex numbers; in fact, $\begin{bmatrix} a & b \\ -b & a \end{bmatrix}$ is like $a + bi$, the identity matrix is like 1, and $R_{90°}$ is like $i$, the square root of $-1$. This is actually a concrete way of seeing that the complex numbers exist.)

**Problem F-17.** In $\mathbf{R}^3$ (only), show that if $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors with the same cross-product relations as $\mathbf{i}, \mathbf{j}, \mathbf{k}$, i.e., $\mathbf{u} \times \mathbf{v} = \mathbf{w}$, $\mathbf{v} \times \mathbf{w} = \mathbf{u}$, and $\mathbf{w} \times \mathbf{v} = \mathbf{u}$, then either $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are orthonormal or else they are all zero vectors.

(Method: Say why they are perpendicular to each other, then write down the relations between their lengths, taking into account that $\sin\theta = 1$ in each cross product. If the lengths are $a, b, c$ respectively, you should get relations $c = ab, bc = a, ca = b$. Substituting one into another you should be able to get $a^3 = a$, etc. What are the solutions?)

**Problem F-18.** (a) Show that a nonzero $3 \times 3$ matrix $P$ is a rotation if and only if the rows of $P$ have the same cross-product relations as $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$; for example, row 1 $\times$ row 2 $=$ row 3. (b) What about such relations between columns?

(Method: Work in terms of the corresponding transformation $T(\mathbf{x}) = \mathbf{x}P$. Remember that the rows of $P$ are the images of the standard basis vectors, or in other words are $T(\mathbf{i})$, $T(\mathbf{j})$, $T(\mathbf{k})$. For $\Rightarrow$ use the fact that $P$ is orthogonal so $P$ preserves the geometric ingredients needed to determine cross products. For $\Leftarrow$, you may quote Problem F-17. Why is orientation preserved rather than reversed?)

**Problem F-19.** In $\mathbf{R}^3$, $R_{90^\circ}^{x\to y} R_{90^\circ}^{y\to z}$ takes points on the $x$-axis to points on the $z$-axis.

(a) Does it equal $R_{90^\circ}^{x\to z}$? (Show your work.)

(b) Is it a rotation matrix? (Give reason.)

(c) Find an axis. (Method: The points on the axis are fixed points. In other words, a point $(a, b, c)$ lies on the axis of the rotation given by a matrix $P$ when $(a, b, c)P = (a, b, c)$.)

**Problem F-20.** (a) Let $T$ be a homogeneous linear transformation that is a rotation in $\mathbf{R}^3$. Explain: The axis of $T$ is the eigenspace for the eigenvalue 1, provided that $T$ is not the identity transformation.

(Recall that if $T$ is a homogeneous linear transformation and $\mathbf{x}$ is a nonzero vector such that $T(\mathbf{x}) = \lambda\mathbf{x}$ for some scalar $\lambda$, then $\mathbf{x}$ is an *eigenvector* of $T$ for the *eigenvalue* $\lambda$. The *eigenspace* of $\lambda$ is $\{\mathbf{x} \in \mathbf{R}^3 | T(\mathbf{x}) = \lambda\mathbf{x}\}$, which is a subspace. In other words, the eigenspace for $\lambda$ consists of all eigenvectors and the zero vector. It makes sense to ask about the eigenspace for $\lambda$ even if $\lambda$ is not an eigenvalue, but in that case the eigenspace is simply the subspace consisting of the zero vector alone.)

(b) If $T$ *is* the identity transformation on $\mathbf{R}^3$, what is the eigenspace for the eigenvalue 1?

(c) What can be said in $\mathbf{R}^2$?

**Problem F-21.** (a) The matrix $M = \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} & \frac{2}{3} \end{bmatrix}$ is a rotation in $\mathbf{R}^3$. Find its axis by finding a nonzero eigenvector for the eigenvalue 1. (See Problem F-20.)

A short way to do this is as follows: Consider a nonzero vector $\mathbf{x}$ along the axis of $M$. Thus $\mathbf{x}M = \mathbf{x}$, or equivalently, $\mathbf{x}(M - I) = \mathbf{0}$. Since a row vector times a matrix is really three dot products, this says that $\mathbf{x}$ is perpendicular to all three columns of $M - I$. So find the cross product of the first two columns of $M - I$.

(b) The short method works for $M$, but what if, say, the first column of $M - I$ were the zero vector? Show that in that case, $\mathbf{i}$ would be along the axis of rotation, so it would not be necessary even to use the short method.

**Problem F-22.** This problem shows that in $\mathbf{R}^3$ every rotation matrix $P$ has an axis, i.e., a line through the origin consisting of fixed points (points $\mathbf{x}$ with $\mathbf{x}P = \mathbf{x}$). For example, if you take a new basketball out of its box, dribble it around, and put it back, its new position is rotated compared to the old, so according to this problem, there will be some axis for this rotation; the two opposite points on the basketball that are on this axis will be in the same place they were before you opened the box.

By Problem F-20, all you have to do is to show somehow that 1 is an eigenvalue of $P$. Thus a statement about rotations that seemed to be geometrical really turns out to be true for algebraic reasons. Some useful facts about eigenvalues are given below.

(a) Show that the list of the three eigenvalues $\lambda_1, \lambda_2, \lambda_3$ of $P$ is the same as the list of their reciprocals (not necessarily in the same order).

(b) Show that 1 is an eigenvalue of $P$. (Method: There are several conceivable ways in which the eigenvalues could match up with the reciprocals. For each way, to show that one of the eigenvalues must be 1.)

Useful facts: If $\mathbf{x}$ is an eigenvector for some eigenvalue $\lambda$, so are all nonzero scalar multiples of $\mathbf{x}$. A matrix and its transpose have the same characteristic polynomial and so have the same eigenvalues; the eigenvalues of the inverse of a matrix are the reciprocals of the eigenvalues of the matrix; the product of the eigenvalues of a matrix equals the determinant; the determinant of a rotation matrix is 1. (Here eigenvalues are listed more than once if necessary; for example, if a matrix has characteristic polynomial $(\lambda + 1)^2(\lambda - 1)$, we would say its eigenvalues are $-1, -1, 1$. In this problem the eigenvalues we are talking about could actually be complex numbers, but you don't need to worry about that in doing the problem, since if 1 is an eigenvalue then there is a corresponding real eigenvector.)

**Problem F-23.** Show that in $\mathbf{R}^3$, every rotation matrix $P$ has an invariant plane through the origin, in other words, a 2-dimensional subspace $S$ such that for all $\mathbf{x} \in S$ you have $\mathbf{x}P \in S$. (You may use the result of Problem F-22. Notice that to say $S$ is invariant is a weaker statement than to say that every point of $S$ is fixed.)

**Problem F-24.** (a) Does the vector $(1, 1, 1)$ make a $45°$ angle with the $z$-axis?

(b) Find a rotation matrix in $\mathbf{R}^3$ that takes a point on the $z$-axis (other than the origin) to a point on the line through the origin and $(1, 1, 1)$. (One way: First rotate with points on the $z$-axis going towards points on the $x$-axis and then rotate with points on the $x$-axis going towards points on the $y$-axis. But by what angles? You may leave your answer as a product of matrices, each with explicit entries.)

(c) Find another answer to (b). Are there still more answers? (Method: If $P$ was your answer, then $R^{x \to y}_{90°} P$ is another answer. Why? Can you generalize this idea?)

**Problem F-25.** Find the entries of a rotation matrix that gives a rotation in $\mathbf{R}^3$ of $90°$ about an axis through the origin and $(1, 1, 1)$, counterclockwise as seen from $(1, 1, 1)$ looking toward the origin.

(Method: Use a three-step idea of moving the axis to a simpler position (the $z$-axis), doing an easier rotation, and moving back. Thus the answer is a product of three rotation matrices. The third one rotates the $z$-axis to the given axis, the middle one is a rotation about the $z$-axis by the given angle, and the first is the inverse of the third. You may use the result of (b) from Problem F-24, so that two of your matrices are themselves already given as products.)

**Problem F-26.** In three dimensions, it is also possible to define what it means for a homogeneous linear transformation $T$ to be a "reflection in a line" $L$, where $L$ is a line through the origin. (a) Invent such a definition. (b) What will the eigenvectors and eigenvalues of $T$ be? (c) Such a transformation is really a rotation. What rotation?

**Problem F-27.** (a) Show that the product of two reflection matrices is a rotation matrix.

(b) In $\mathbf{R}^2$, what rotation is obtained by first reflecting with the $x$-axis as a mirror and then reflecting with the $y$-axis as a mirror?

(c) In $\mathbf{R}^3$, what rotation is obtained by first reflecting with the $x, z$-plane as a mirror and then reflecting with the $y, z$-plane as a mirror? (Answer with a matrix and describe the axis of the rotation.)

**Problem F-28.** (a) Let $M = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, a reflection in the $x$-axis. Show that $MR_\theta M = R_{-\theta}$, both by a calculation and by describing what the effect is if you do the left side to an object such as a piece of paper. (For clarity, imagine using paper with printing on it.)

(b) Show that $R_{-\theta}M = MR_\theta$. (Use (a), or calculate directly.)

(c) Show that the matrix of a reflection in $\mathbf{R}^2$ whose mirror line makes an angle $\theta$ with the $x$-axis is $MR_{2\theta}$. (Start from the "three-step method.")

**Problem F-29.** Show that in $\mathbf{R}^3$, these matrices are orthogonal, but are neither rotations nor reflections (in a plane):

(a) $-I$ (where $I$ is the $3 \times 3$ identity matrix);

(b) the product of $R_{90°}^{x \to y}$ and a reflection with the $x, y$-plane as mirror.

(Method: The fixed points of a reflection in $\mathbf{R}^3$ are the points of the mirror, which should be a plane. Find the fixed points of these matrices and see if they do form a plane.)

(c) In $\mathbf{R}^2$, is $-I$ a reflection or rotation, or neither? If it is a reflection, what is the mirror? If it is a rotation, by what angle? If neither, can it be expressed as in (b)?

(*Note.* Sometimes $-I$ is thought of as a "reflection in a point" (the origin), but for us a reflection means a reflection in a mirror plane in $\mathbf{R}^3$ or a mirror line in $\mathbf{R}^2$.)

**Problem F-30.** Consider the h.l.t. $T : \mathbf{R}^4 \to \mathbf{R}^4$ whose matrix is $P = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$. Notice that $T$ permutes the axes, taking each one to the next cyclically. $P$ is an orthogonal matrix. Is $T$ a rotation, a reflection (in a three-dimensional "hyperplane"), or neither?

**Problem F-31.** A *group of matrices* is a set of matrices that contains $I$, is "closed under taking products", and is "closed under taking inverses". In other words, the product of two matrices in the group is itself in the group and the inverse of a matrix in the group is in the group. According to Corollary 1.3, the orthogonal $n \times n$ matrices form a group, called the *orthogonal group* $\mathrm{O}(n)$. According to Proposition 2.2, the $n \times n$ rotation matrices form a group, called the *rotation group* $\mathrm{SO}(n)$ ("special orthogonal"). Do reflections form a group of matrices?

**Problem F-32.** Find the matrix of a reflection in $\mathbf{R}^2$ whose mirror line is the line through the origin 60° counterclockwise from the $x$ axis. (Method #1: Express your answer as a product of three matrices: The matrix that rotates the mirror line to the $x$-axis, the reflection matrix whose mirror is the $x$-axis, and the matrix that undoes the first rotation. Method #2: Apply the formula from (6) of §4.)

**Problem F-33.** There are only two $2 \times 2$ rotation matrices that have real eigenvalues. (a) Which ones? (b) Why are there no others? (Give some explanation.)

**Problem F-34.** Recall that the latitude of a point on the earth is the angle from the equator to the point, as measured from the center of the earth, and the longitude of a point on the earth is the angle from Greenwich ["gren'itch"], England to the point, as measured looking from above the north pole. (Angles can be negative if appropriate.) It is handy to imagine the earth as a unit sphere in $\mathbf{R}^3$, with the origin at the center of the earth, the positive $x$-axis going through the point with zero latitude and zero longitude, and the positive $z$-axis going through the north pole.

(a) Describe a rotation matrix that would take $(1, 0, 0)$ to the point with latitude 30° and longitude 50°. (You may express your answer as the product of two specific rotation matrices.)

(b) Find a formula for the Cartesian coordinates $(x, y, z)$ of the point on the earth with latitude $\theta$ and longitude $\phi$. (Method: Use the idea of (a) and multiply out. For example, you should get $x = \cos\theta\cos\phi$.)

**Problem F-35.** Invent a method of describing a rotation in $\mathbf{R}^3$ uniquely by using three numbers. (You may use the fact that the rotation has an axis.)

**Problem F-36.** Can every $3 \times 3$ rotation matrix be obtained in the form $R_\theta^{x \to y} R_\psi^{y \to z}$ for suitable $\theta$ and $\psi$? (Method: If the answer were "yes", that would mean three-dimensional rotations could be described with just two numbers, which sounds unlikely in view of the preceding problem. Show that in fact, $R_{90°}^{x \to z}$ cannot be obtained in the specified form, by considering what happens to a vector on the $z$-axis.)

**Problem F-37.** A puzzle: Make a rotation matrix by filling in $\begin{bmatrix} .60 & .48 & \sqcup \\ -.80 & \sqcup & \sqcup \\ \sqcup & \sqcup & \sqcup \end{bmatrix}$.

(Give reasoning. As in Problem F-35 and Problem F-36, notice that there are three numbers of information; the answer is almost unique but not quite.)

**Problem F-38.** (a) In $\mathbf{R}^2$, is the product of three reflections always a reflection? Give a reason for your answer (an explanation if it's true, or a counterexample if it's not).

(b) Same question for $\mathbf{R}^3$.

**Problem F-39.** Show that $R_\theta - R_{-\theta}$ is a scalar multiple of $R_{90^\circ}$.

**Problem F-40.** In $\mathbf{R}^3$, find the matrix of the reflection in the mirror $x + y + z = 0$.

**Problem F-41.** For a unit vector $\mathbf{n}$ in $\mathbf{R}^3$, let $M = I - 2\mathbf{n}^t\mathbf{n}$, as in (6) of §4, and let $T(\mathbf{v}) = \mathbf{v}M$.

(a) Check directly that if $\mathbf{v}$ is a scalar multiple of $\mathbf{n}$, then $T(\mathbf{v}) = -\mathbf{v}$.

(b) Check directly that if $\mathbf{v}$ is perpendicular to $\mathbf{n}$, i.e., lies in the plane through the origin with normal $\mathbf{n}$, then $T(\mathbf{v}) = \mathbf{v}$.

(c) If you do a reflection twice, you should get the vector you started with. Show directly that $T(T(\mathbf{v})) = \mathbf{v}$, or equivalently, $M^2 = I$.

**Problem F-42.** In $\mathbf{R}^2$, use (6) of §4 to find the matrix of the reflection whose mirror line goes through the origin and $(4, 3)$. (You will need to find $\mathbf{N}$.)

**Problem F-43.** (a) For a reflection $T$ in $\mathbf{R}^3$ whose mirror plane has unit normal $\mathbf{n}$, show that $T(\mathbf{x}) = \mathbf{x} - 2(\mathbf{x} \cdot \mathbf{n})\mathbf{n}$.

(Method: For any vector $\mathbf{x}$, write

(*) $\mathbf{x} = \mathbf{x}_\perp + \mathbf{x}_{||}$,

where $\mathbf{x}_\perp$ is perpendicular to the mirror plane and $\mathbf{x}_{||}$ is parallel to the mirror plane. You know that the length of $\mathbf{x}_\perp$ is $\mathbf{x} \cdot \mathbf{n}$, so $\mathbf{x}_\perp = (\mathbf{x} \cdot \mathbf{n})\mathbf{n}$. Then $\mathbf{x}_{||}$ is simply $\mathbf{x}$ minus this. Now use the observation that $T$ leaves $\mathbf{x}_\perp$ fixed but negates $\mathbf{x}_{||}$. What happens when you apply $T$ to (*)?)

(b) Derive (6) of §4. (Method: Starting from (a), use the observation that a dot product $\mathbf{u} \cdot \mathbf{v}$ can be expressed as $\mathbf{u} \cdot \mathbf{v}^t$, the product of a row vector and a column vector. Also, a product of three matrices can be associated in either order. Write $\mathbf{x} = \mathbf{x}I$ and see if you can get $T(\mathbf{x}) = \mathbf{x}(\cdots)$.)

**Problem F-44.** Find the matrix of the reflection in $\mathbf{R}^2$ whose mirror line is the line through the origin slanted at 30°, by using three methods: (a) The "three-step" method, as in lecture; (b) the method of the fact (6) of §4, and (c) the method of finding $MR_{2\theta}$, as discussed in one of the problems above. Your three answers should agree.

**Problem F-45.** Recall that a *rational number* is the ratio of two integers, i.e., is expressible as a fraction. Thus $\frac{5}{3}$, $-0.23$, and $7$ $(= \frac{7}{1})$ are rational. In contrast, $\sqrt{2}$, $\pi$, and $e$ are *irrational*. A vector or matrix is said to be rational if all entries are rational, and is said to be irrational otherwise. For a plane $ax + by + cz = 0$ with $a, b, c$ rational, usually the unit normal to that plane will be irrational. Show that, nevertheless, the matrix of the reflection in $\mathbf{R}^3$ with that plane as mirror plane is necessarily rational. (Method: Does any formula for this matrix involve only operations which for rational numbers give rational answers, such as addition, rather than operations such as taking square roots, which don't?)

**Problem F-46.** The matrix of a reflection in $\mathbf{R}^3$ (with mirror plane through the origin) will always have eigenvalues $1, 1, -1$. (a) Explain why. (b) What is its characteristic polynomial?

**Problem F-47.** As mentioned in Problem F-21, the matrix
$M = \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} & \frac{2}{3} \end{bmatrix}$ is a rotation in $\mathbf{R}^3$. By what angle?
(Method: Find a nonzero vector $\mathbf{y}$ perpendicular to the axis of $M$ and then find the angle between $\mathbf{y}$ and $\mathbf{y}M$. Such a vector $\mathbf{y}$ can be found without actually finding the axis, as follows: As in Problem F-21, the columns of $M - I$ are perpendicular to the axis of the rotation, so just let $\mathbf{y}$ be the first column of $M - I$. In general, if the first column of $M - I$ were $\mathbf{0}$ we'd have to choose another column, but that's not the case in this example.)

**Problem F-48.** Show that a $3 \times 3$ orthogonal matrix $P$ of determinant $-1$ is the product of a rotation $R$ and a reflection $M$ in a mirror perpendicular to the axis of $R$.

(Method: Since the characteristic polynomial of $P$ is cubic, $P$ has a real eigenvalue, and since $P$ gives a rigid transformation that eigenvalue is either $+1$ or $-1$. In other words, $P$ either has a fixed vector or a vector that is negated. Take $M$ to have mirror with this vector as normal. Consider the two cases separately.)