Solutions to Midterm #1

The distribution of scores will be accessible on the web. Even if you got a low score this time, as you get a better idea of what is expected in writing down reasoning you should be able to do much better on the second midterm and final. Anyone who gets a solid grade on the final and has done the homework can get a good grade in the course. Most people are doing OK on quizzes and homework, so they should be able to learn to do well on exams.

Although letter grades are not recorded for exams, a rough idea of the grade equivalent of scores would be A–/A 40-50, B- to B+ 30-39, C- to C+ 20-29. This means the average grade was low this time, but as people get better at writing the "proofs to know" the average grade should become much higher by the end of the course.

In terms of writing down a proof, it is expected that the answer should mention the key points of the proof in the correct relationship, e.g., not having implications in the wrong direction. It is also expected that there should not be nonsense, e.g., "Each vector is linearly independent", "the vector space has n vectors", $v_i \cong F^n$, etc.

1. (a) We know $\mathbb{R}^3 \cong \text{Pols}(\mathbb{R}, 2)$ with $(a, b, c) \mapsto a + bx + cx^2$, so we can move the problem over to $\mathbb{R}^3$, where the question becomes whether the vectors $(2, 2, 1)$, $(2, 3, 3)$, $(10, 13, 11)$ are linearly independent. Make M with these as its rows. $M \rightsquigarrow \ldots \rightsquigarrow \begin{bmatrix} 1 & 0 & -\frac{3}{2} \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$, so M has rank 2 and the vectors are linearly dependent. Therefore the original polynomials are linearly dependent.

(b) For convenience, let's write this as $rM_1 + sM_2 + tM_3 + uM_4 = \mathcal{O}$.

Method #1: Just notice $M_2 - M_1 = M_3 - M_2$, which gives a linear relation $M_1 - 2M_2 + M_3 = \mathcal{O}$. Shifting over one position we also get $M_2 - 2M_3 + M_4 = \mathcal{O}$. The coefficients involved in these two relations are $(r, s, t, u) = (1, -2, 1, 0)$ and $(0, 1, -2, 1)$, not proportional.

Method #2: Use $\mathbb{R}^4 \cong \text{Mats}(\mathbb{R}, 2, 2)$ by $\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \to \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, so we need to find the linear relations between columns of $M = \begin{bmatrix} 1 & 3 & 5 & 7 \\ 2 & 4 & 6 & 8 \\ 3 & 5 & 7 & 9 \\ 4 & 6 & 8 & 10 \end{bmatrix}$, which are given by the null space of M.

Row reducing, $M \rightsquigarrow \ldots \rightsquigarrow E = \begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. The general solution to $E \begin{bmatrix} r \\ s \\ t \\ u \end{bmatrix} = \mathbf{0}$ is $\begin{bmatrix} r \\ s \\ t \\ u \end{bmatrix} = t \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix} + u \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix}$. Therefore two independent choices of coefficients for the linear relations are $(r, s, t, u) = (1, -2, 1, 0)$ and $(2, -3, 0, 1)$.

2. To get $W_1 \cap W_2$, we simply put together the two sets of equations and take the solution space. So we want a basis for the null space of $M = \begin{bmatrix} 1 & 1 & 1 \\ 3 & 2 & 1 \\ -1 & 1 & 3 \\ 3 & 1 & -1 \end{bmatrix}$. We get $M \rightsquigarrow \ldots \rightsquigarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, for which a basis for the null space is $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$.

3. (a) We use these facts: (i) If V has a basis of m elements then $V \cong F^m$; (ii) in F^m, any $m+1$ or more vectors are linearly dependent.

If V has a basis of m elements and a basis of n elements, then by (i), $V \cong F^m$ and $V \cong F^n$ so $F^m \cong F^n$. Suppose $m < n$. Then the standard basis vectors of F^n correspond to a list of n linearly independent vectors in F^m, which contradicts (ii). If $m > n$ we get a similar contradiction in which m and n are switched. So the only possibility is that $m = n$.

(b) Suppose $v_1, \ldots, v_m$ is a minimal spanning set for V. If they were linearly independent, then some v_i would be in the span of the others, so if we were to omit it the span of the new list would still be all of V. But then the original list was not minimal, a contradiction. Therefore they are linearly dependent.

4. Let A be any matrix with entries in some field, and suppose E is the row-reduced echelon form of A. We know that row reduction doesn't change linear relations between columns, so E has the same linear relations as A. Using linear relations between columns we can tell whether a given column is or is not a linear combination of other specified columns, and if it is, then with what coefficients. The pivot columns of E are those that are not in the span of the preceding columns; their entries are determined, since they are the same as some or all of the columns of an indentity matrix, in order. Each non-pivot column is in the span of the preceding pivot columns, and its entries consist of the coefficients used; therefore the entries of the non-pivot columns are determined by the linear relations. All together, then, the entries of E are completely determined by A. In other words, the row-reduced echelon form of A is unique.

5. (a) False: A and A^t have the same rank, since row rank = column rank and rows of A are columns of A^t.

(b) False: if $F = \mathrm{GF}(2) = \mathbb{Z}_2$ then in F^2 with $v = (1, 1)$ we have $v + v = \mathbf{0}$.

(c) True: Using the theorem that $\dim(W_1 + W_2) + \dim(W_1 \cap W_2) = \dim W_1 + \dim W_2$ we have $4 + ? = 2 + 2$, so the $\dim W_1 \cap W_2 = 2$.

(d) True: Two vectors are linearly dependent when one is a scalar multiple of the other. Since the only scalars are 0 and 1, if both vectors are nonzero they would have to be equal, which they are not.