How to find bases in various situations

Recall that a vector given as an n-tuple can also be written as a row vector ($1 \times n$ matrix) or column vector ($n \times 1$ matrix). For example, $(3, 2, 5)$ can also be written as $\begin{bmatrix} 3 & 2 & 5 \end{bmatrix}$ or $\begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix}$. In this course, column vectors are usually the appropriate choice, as we'll discuss.

1. Some useful facts

1. Row reduction doesn't change

   - the row space of a matrix,
   - the linear relations between columns of a matrix,
   - the null space of a matrix.

   (Careful—row reduction *can* change the column space, and usually does.)

2. If you have several vectors such that each one has 1 in some entry for which the others have 0, then they are linearly independent.

   Example: $(1, 2, 0, 3, 0)$, $(0, 4, 1, 2, 0)$, $(0, 3, 0, -2, 1)$, or the same thing with row vectors or column vectors.

3. A linear combination of column vectors is the same as the matrix made from the columns times the column vector of coefficients.

   Example:

$$x \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + y \begin{bmatrix} 2 \\ 4 \\ 2 \end{bmatrix} + z \begin{bmatrix} 1 \\ 3 \\ 3 \end{bmatrix} + s \begin{bmatrix} 2 \\ 7 \\ 8 \end{bmatrix} + t \begin{bmatrix} 3 \\ 8 \\ 8 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 & 2 & 3 \\ 2 & 4 & 3 & 7 & 8 \\ 1 & 2 & 3 & 8 & 8 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ s \\ t \end{bmatrix}$$

   Advice: When you see a linear combination of column vectors like this, you should *always* think of the matrix form also.

2. The row space of a matrix

Problem. Find a basis for the row space of

$$M = \begin{bmatrix} 1 & 2 & 1 & 2 & 3 \\ 2 & 4 & 3 & 7 & 8 \\ 1 & 2 & 3 & 8 & 8 \end{bmatrix}.$$

Method. Row-reduce and take the nonzero rows. They have the same span as the original rows and are also linearly independent.

Note. This is also a way to find a basis for the span of a list of vectors: Make a matrix with those vectors as rows and row reduce. (See also section 6.)

3. The solution space to a set of homogeneous linear equations

Problem. Find a basis for the solution space of

$$\begin{array}{rcrcrcrcrcl} x & + & 2y & + & z & + & 2s & + & 3t & = & 0 \\ 2x & + & 4y & + & 3z & + & 7s & + & 8t & = & 0 \\ x & + & 2y & + & 3z & + & 8s & + & 8t & = & 0 \end{array}$$

Solution. Don't bother to augment M with a constant column, since it would be all 0's and would stay that way during row reduction. Just row-reduce M itself, getting

$$E = \begin{bmatrix} 1 & 2 & 0 & -1 & 0 \\ 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

The non-pivot variables are y and s. The general solution is

$$\begin{array}{rcrcr} x & = & -2y & + & s \\ y & = & y & & \\ z & = & & & -3s \\ s & = & & & s \\ t & = & 0 & & \end{array}, \quad \text{or equivalently,} \quad \begin{bmatrix} x \\ y \\ z \\ s \\ t \end{bmatrix} = y \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 1 \\ 0 \\ -3 \\ 1 \\ 0 \end{bmatrix}.$$

The basis, then, consists of these two column vectors.

4. The null space of a matrix

Problem. Find a basis for the null space of

$$M = \begin{bmatrix} 1 & 2 & 1 & 2 & 3 \\ 2 & 4 & 3 & 7 & 8 \\ 1 & 2 & 3 & 8 & 8 \end{bmatrix}.$$

In other words, find a basis for the vector solutions of $M\mathbf{x} = \mathbf{0}$.

Method. This is the same as finding a basis for the solution space of the homogeneous linear equations with coefficient matrix M.

5. Linear relations between vectors

Problem. Find a basis for the linear relations between these vectors:

$$\begin{bmatrix}1\\2\\1\end{bmatrix}, \begin{bmatrix}2\\4\\2\end{bmatrix}, \begin{bmatrix}1\\3\\3\end{bmatrix}, \begin{bmatrix}2\\7\\8\end{bmatrix}, \begin{bmatrix}3\\8\\8\end{bmatrix}$$

Method. A linear relation means a linear combination of these vectors that equals $\mathbf{0}$. Treat the list of coefficients as a vector itself. In other words, we are looking for a basis for the space of coefficients

$$\begin{bmatrix}x\\y\\z\\s\\t\end{bmatrix} \text{ such that } x\begin{bmatrix}1\\2\\1\end{bmatrix} + y\begin{bmatrix}2\\4\\2\end{bmatrix} + z\begin{bmatrix}1\\3\\3\end{bmatrix} + s\begin{bmatrix}2\\7\\8\end{bmatrix} + t\begin{bmatrix}3\\8\\8\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

As in Section 1, this is the same as saying $M\mathbf{x} = \mathbf{0}$, where M is the matrix with these columns, so the answer is a basis for the null space of M.

6. Basis within a spanning set

Problem. Among the list of vectors

$$\begin{bmatrix}1\\2\\1\end{bmatrix}, \begin{bmatrix}2\\4\\2\end{bmatrix}, \begin{bmatrix}1\\3\\3\end{bmatrix}, \begin{bmatrix}2\\7\\8\end{bmatrix}, \begin{bmatrix}3\\8\\8\end{bmatrix}$$

choose some that form a basis for the span of all five.

Solution. Form the matrix M with these as columns and row reduce; let's say E is the row-reduced matrix. In E, notice that the non-pivot columns are in the span of the pivot columns. Since M has the same relations between columns as E, the same pivot columns in M are the ones you want for your basis.

Here $E = \begin{bmatrix}1&2&0&-1&0\\0&0&1&3&0\\0&0&0&0&1\end{bmatrix}$. The pivot columns are columns 1, 3, and 5.

So in the original list of vectors, the first, third, and fifth make a basis.

Note. If you are asked simply to find a basis for the span of a list of vectors, with no method specified, you have a choice:

(1) Make these the *rows* of a matrix and row-reduce to get a basis for the row space, as in Section 2, or (2) rewrite these vectors as column vectors and select some to be a basis, as in this section.

7. Matrix with given null space

Problem. Find a matrix M whose null space is the span of

$$\begin{bmatrix} 1 \\ 2 \\ 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 3 \\ 7 \\ 8 \end{bmatrix}, \text{ and } \begin{bmatrix} 1 \\ 2 \\ 3 \\ 8 \\ 8 \end{bmatrix}.$$

Discussion. M should have five columns; we don't yet know how many rows. Concentrate on one row of M; since it's unknown, call it $\mathbf{x} = \begin{bmatrix} x & y & z & s & t \end{bmatrix}$. When we multiply it by any of the three given column vectors we are supposed to get 0. Since $\mathbf{x}$ times a matrix equals $\mathbf{x}$ times each column of the matrix, if we let A be the matrix with the given vectors as columns we have $\mathbf{x}A = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$. This equation describes the space of all row vectors eligible to be a row of M. So we should choose M to have rows that are a basis for this space. How? Just transpose to get a more familiar kind of problem, a matrix times an unknown column vector: $A^t\mathbf{x}^t = \mathbf{0}$. (Here we used the idea $(BC)^t = C^tB^t$.) Now we know what to do:

Method. Make A with the given vectors as columns, find a basis for the null space of A^t, and use those basis vectors as the rows of M. (In doing that we're transposing the basis vectors to make them row vectors.)

8. Basis for a sum of subspaces

Problem. In $\mathbb{R}^3$, if W_1 is spanned by $(1,2,1)$, $(2,4,2)$, $(1,3,3)$ and W_2 is spanned by $(2,7,8)$, and $(3,8,8)$, find a basis for $W_1 + W_2$.

Method. Just put these lists together and find a basis, as in Section 2 or Section 6.

9. Basis for an intersection of subspaces

Problem. Find a basis for the intersection in $\mathbb{R}^3$ of the subspace W_1 spanned by $(1,2,1)$, $(2,4,2)$, and $(1,3,3)$ and the subspace W_2 spanned by $(2,7,8)$ and $(3,8,8)$.

Method. We can't use basis elements from W_1 and W_2, since they are almost certainly outside the intersection. Instead, we can use an indirect method: Find homogeneous linear equations whose solution space is W_1 (or in other words, find the rows of a matrix whose null space is W_1, as in Section 7), and similarly for W_2. Taking both sets of equations together, the solution space will be $W_1 \cap W_2$. Then find a basis for this solution space.