# Assignment #3

**Quiz 3** in discussion section, **Tuesday, October 16:**

You will be asked to do some problem like E-1 and p. 49 Ex. 9, but with messier coefficients, two ways: (a) directly, (b) using isomorphism. For sample proofs, see the on-line notes for 2-W.

**Assignment** due in lecture **Wednesday, October 17:**

| where | Do but don't hand in | Hand in |
|---|---|---|
| p. 15 | 2,3 | |
| p. 49 | | 10, 13 |
| p. 66 | 3 | |
| G | G-5 | G-1, G-2, G-3, G-4, G-6, G-7, G-8, G-9 |

**Problem G-1.** In the first Theorem on Handout F, show

(a) (4) $\Rightarrow$ (1);

(b) (5) $\Rightarrow$ (1).

**Problem G-2.** For a function $f : S \to T$ between sets, a function $g : T \to S$ is a *left inverse* of $f$ if $g(f(s)) = s$ for all $s \in S$, or a *right inverse* if $f(g(t)) = t$ for all $t \in T$.

Show that $f$ has a left inverse $\Leftrightarrow$ $f$ is one-to-one, and a right inverse $\Leftrightarrow$ $f$ is onto. (In both cases there may be some choice involved in making $g$, so $g$ is not necessarily unique.)

*Note.* If $f$ is both one-to-one and onto, then $f$ has a unique "two-sided" inverse $g$, which we call $f^{-1}$, as in E-5.

**Problem G-3.** Find the rank of the $10 \times 10$ matrices consisting of

(a) the addition table for integers $0, \ldots, 9$: $\begin{bmatrix} 0 & 1 & \cdots & 9 \\ 1 & 2 & \cdots & 10 \\ \cdots & \cdots & \cdots & \cdots \\ 9 & 10 & \cdots & 18 \end{bmatrix}$

(b) the multiplication table for integers $0, \ldots, 9$: $\begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 9 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 9 & \cdots & 81 \end{bmatrix}$

**Problem G-4.** Let $F = GF(2) = \mathbb{Z}_2 = \{0, 1\}$.

(a) Show that the vector space $F^2$ has five subspaces. (Don't forget the subspace $\{\mathbf{0}\}$.)

(b) Show that the vector space $F^3$ has sixteen subspaces—even though the vector space itself has only eight elements! (Describe subspaces using bases.)

**Problem G-5.** If you take all square matrices of a fixed size, say $2 \times 2$, with entries in some field $F$, you can both add them and multiply them. Which of the field properties described on pages 1 and 2 hold? (Not all do! No proofs required.)

**Problem G-6.** If you take just *some* $2 \times 2$ matrices with entries in a field $F$, they *might* form a field themselves. The first requirement is that your set of matrices should be closed under addition and multiplication.

Show that the field $\mathbb{C}$ of complex numbers is isomorphic to the set of all real matrices of the form $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$.

(This means isomorphic as fields. Method: Let $S$ be this set of matrices. Define a function $\phi : \mathbb{C} \to S$ in some reasonable way, saying $\phi(a + bi) = \cdots$. (Mathematicians in the U.S. usually pronounce $\phi$ as "fee" rather than "fi".) Make sure $\phi$ is one-to-one and onto and preserves the relevant operations. Although you could check directly that $S$ is closed under matrix $+$ and $\cdot$, this actually follows automatically from the fact that $\phi$ preserves operations; for example $M_1, M_2 \in S \Rightarrow M_1 M_2 = \phi(z_1)\phi(z_2) = \phi(z_1 z_2) \in S$.)

**Problem G-7.** In $\mathbb{Z}_{13}$, find the multiplicative inverse of each nonzero element.

The next two problems are warm-ups for future problems.

**Problem G-8.** From a deck of playing cards, take Ace (= 1), 2, 3, 4 of all four suits. Find an arrangement of these cards as follows: The cards are laid in a $4 \times 4$ square. Each row of the square has four different suits and four different numbers. So does each column.

**Problem G-9.** In statistical experiments involving, say, plants, it is helpful to be able to overlap various conditions in an efficient but uniform way. For example, show how to take a set of seven plants and designate seven blocks of three plants each so that (i) any two plants are in exactly one block and (ii) any two blocks have exactly one plant in common. (Then one block could be "extra sun", one could be "extra water", one could be "rich soil", etc.)