Vector Spaces

1. The idea

Let's stick to real vector spaces for a moment, in other words, vector spaces with scalars in $\mathbb{R}$.

Consider the space $\mathbb{R}^n$ with the operations of addition and scalar multiplication. A vector space "over $\mathbb{R}$" is intended to be any set with an addition-like operation and some scalar-multiplication-like operation obeying all the laws that $\mathbb{R}^n$ does, for all n.

The trouble is that the operations of $\mathbb{R}^n$ obey infinitely many laws, if you consider not only familiar ones such as $v + w = w + v$, but also messier ones such as $3v + (3w - u) = 3(v + w) - u$.

Some laws can be proved from others. People have analyzed the situation and found a list of basic laws from which all the other laws can be proved. The definition of a vector space simply lists the basic laws.

The same laws work fine for scalars in any field.

2. The definition

Definition. A *vector space* over a field F is a set V with

(i) a binary operation $+$, an element $\mathbf{0}$, and for each $v \in V$, an element $-v$,

(ii) an operation of "multiplication by scalars" so that $r \in F$ and $v \in V$ give an element $rv \in V$, such that

(a)	$v + w = w + v$	(commutative law for addition)
(b)	$v + (w + u) = (v + w) + u$	(associative law for addition)
(c)	$v + \mathbf{0} = v$	($\mathbf{0}$ is a neutral element for addition)
(d)	$v + (-v) = \mathbf{0}$	(each element has a negative)
(e)	$1v = v$	(1 is neutral for mult. by scalars)
(f)	$(rs)v = r(sv)$	(mult. in F versus mult. by scalars)
(g)	$r(v + w) = rv + rw$	(mult. by scalars versus vector addition)
(h)	$(r + s)v = rv + sv$	(mult. by scalars versus addition in F)

3. Properties that can be proved from other definition

(i)	$r\mathbf{0} = \mathbf{0}$	(scalar times the zero vector)
(j)	$0v = \mathbf{0}$	(0 scalar times any vector)
(k)	$(-1)v = -v$	(mult. by -1 is same as negation)

(Proofs are in the text, p. 31.)

4. Notes

1. In the laws, we really mean that the equations are true for all values of $v, w, u \in V$ and all $r, s \in F$.

2. Notice that (a)–(d) are about addition only (an "additive group"), while (e)–(h) involve multiplication by scalars.

3. The text's definition of a vector space mentions F as if it is part of the vector space: "A vector space consists of a field F and a set V such that ... ". That's a little old-fashioned. The difference is only technical.

4. Don't get confused between the zero scalar and the zero vector. I'll usually write the 0 vector as $\mathbf{0}$ or $\vec{0}$. In $\mathbb{R}^2$, for example, these are 0 and $\mathbf{0} = (0, 0)$.

5. The operation minus as mentioned above is "unary": $-v$. However, it is easy to define binary minus by saying $v - w$ is $v + (-w)$.

6. A binary operation is really a function on pairs. Addition is a function $+ : F^2 \to F$, while multiplication by scalars is a function $\cdot : F \times V \to V$.

7. In the course it will eventually become clear why all the laws of $\mathbb{R}^n$ follow from the basic laws listed.

8. These various laws together really say that for algebra with scalars and vectors you can "do what comes naturally". In this course you are free to do algebra on scalars and vectors without explanation—except when we're talking about proving some laws from others!

5. Problems

Problem D-1. Prove these facts, using any laws above:

(a) $rv = \mathbf{0} \Rightarrow r = 0$ or $v = \mathbf{0}$.

(b) $rv = rw$ and $r \neq 0$ imply $v = w$ (cancellation of a scalar).

(c) $rv = sv$ and $v \neq \mathbf{0}$ imply $r = s$ (cancellation of a vector).

Note: (a) is on p. 31.

Problem D-2. Equivalent definitions of a vector space are possible. For instance, since $(-1)v = -v$, we could get away with not mentioning $-v$ in the definition of a vector space. In that case, we would omit (d). However, then we can't prove (j), so we need to make that one of our defining properties.

The problem: Starting from (a)–(c), (e)–(h), and (j) as the "new" definition of a vector space, define what $-v$ means and then *prove* (d), quoting the laws you are using.