Assignment #9

Quiz 9 in discussion section **Tuesday, November 27**:

For a linear transformation $T : V \to W$ between vectors spaces and a subspace S of W, (a) be able to define what $T^{-1}(S)$ means, and (b) be able to prove that $T^{-1}(S)$ is a subspace of V. (See Problem O-3 and its solution.)

Assignment due **Friday, November 30**.

where	Do but don't hand in	Hand in
U	7	
CC	CC-1, CC-6, CC-7, CC-8, CC-9	CC-2, CC-3, CC-4, CC-5, CC-10
DD	DD-1, DD-3, DD-5, DD-7	DD-2, DD-4, DD-6, DD-8, DD-9

Problem CC-1. For the linear transformation $L_A : \mathbf{R}^2 \to \mathbf{R}^2$ with $A = \begin{bmatrix} 3 & -1 \\ -3 & 1 \end{bmatrix}$, sketch (a) the range, (b) the null space, and (c) the inverse image of $\begin{bmatrix} 2 \\ -2 \end{bmatrix}$ [the set of vectors in the domain that go to this vector]. Use two sets of axes. Do a careful job, indicating the scale by "tick marks" on the axes. Label your answers.

Problem CC-2. Let V be the solution space of the DE $y'' = -4y$. Let B be the basis consisting of $\cos 2x, \sin 2x$. If $T : V \to V$ is the differentiation operator, find the matrix of T relative to this basis.

Problem CC-3. (a) Diagonalize the matrix $A = \begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix}$. Be sure to give P and D, but you don't need to find P^{-1}.

(b) Diagonalize A^2. (Again, give P.)

Problem CC-4. (a) Diagonalize $A = \begin{bmatrix} 2 & 3 \\ 6 & 5 \end{bmatrix}$, showing the matrices involved, with explicit entries, including finding a matrix inverse where needed. Show that your answer checks, by multiplying out.

(b) Find a matrix B with $B^3 = A$.

(Method: Diagonalize A using an appropriate P, take a cube root, and "undiagonalize" by doing a similarity back using P^{-1} in place of P.)

Problem CC-5. Show that for an $m \times n$ matrix A and $n \times m$ matrix B over the same field, $\text{trace}(AB) = \text{trace}(BA)$.

Problem CC-6. (a) Show that square matrices A, B of the same size, if at least one of A and B is invertible (i.e., nonsingular) then AB and BA are similar. (Method: Easy matrix manipulation.)

(b) Looking through past homework, find an example to show that (a) might not hold if neither of A and B is invertible. (Method: What if $AB = 0$?)

Problem CC-7. Invent a specific numeric example of each of the following, giving a brief reason in each case:

(a) A 2×2 matrix with eigenvalues whose sum is 1 and whose product is -1.

(b) A 2×2 diagonal matrix that is a rotation but is not the identity matrix.

(c) A matrix A such that $A^2 - A + I = 0$. (Method: Just find a 2×2 matrix A with characteristic polynomial $\lambda^2 - \lambda + 1$. Magically, any matrix acts like a root of its own characteristic polynomial. Check your answer.)

(d) A matrix A and eigenvalue λ so that the eigenspace E_λ has dimension 2.

Problem CC-8. By "permutation" we will always mean a permutation on a finite number of symbols. Recall that a *transposition* means a 2-cycle such as $\begin{pmatrix} 1 & 3 \end{pmatrix}$.

(a) Show that the n-cycle $\begin{pmatrix} 1 & 2, \ldots, n \end{pmatrix}$ is a product of transpositions. In other words, you can achieve an n-cycle by switching two symbols at a time, several times. (Advice: Try this for $n = 3$, then $n = 4$, etc. until you see a general method.)

(b) Show that every permutation is a product of transpositions. (Quote the fact that every permutation is a product of "disjoint" cycles and use (a).)

(c) A permutation σ is said to be *even* if σ is the product of an even number of transpositions, or *odd* if σ is the product of an odd number of transpositions.

It is a fact that any permutation is either even or odd, but not both.

Looking at the multiplication table for S_3 (p. AA 3), say which permutations are even, which are odd, and check that a transposition times an even permutation is odd and vice versa. (It may help to draw a horizontal line and vertical line to split the table into four areas.)

Problem CC-9. (a) For functions $x(t), y(t)$, solve the system of differential equations (DE's)

$$\begin{cases} x' &= 4x & \\ y' &= & -7y \end{cases} \quad \text{with } x(0) = 1,\ y(0) = 2.$$

(Method: Recall that for a single dependent variable the solution of $y' = ky$ is $y = y(0)e^{kt}$.)

(b) Rewrite the problem in matrix form. (Method: $\mathbf{x}' = D\mathbf{x}$ with $\mathbf{x}(0) = ??$. What is D?)

(c) Show that the solution can be expressed in matrix form using a matrix power, as in Problem W-2, except that the initial values become a vector on the right: $\mathbf{x}(t) = ??\mathbf{x}(0)$.

(d) Check the solution directly by taking the derivative of a matrix power that is a function of t, inventing any reasonable rules you need for derivatives of matrix functions of t.

Problem CC-10. Solve the system of DE's $\mathbf{x}' = A\mathbf{x}$ where $A = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}$ and $\mathbf{x}(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, two ways:

(a) by using similarity;

(b) by using a matrix power.

(Methods: For (a) diagonalize A by $P^{-1}AP = D$ and substitute $\mathbf{x} = P\mathbf{z}$, where $\mathbf{z} = \begin{bmatrix} z \\ w \end{bmatrix}$. Use algebra to get the system of DE's $\mathbf{z}' = D\mathbf{z}$ and also find $\mathbf{z}(0)$. Solve this diagonal system and then put things back in terms of $\mathbf{x}$. The solution does not have to be in matrix form.

For (b) just propose a solution and then check it by taking the derivative, inventing any needed rules as in the problem CC-9.)