Solutions to Midterm #2

Questions on grading are welcome, even for one point. Please write any questions on the front of the exam and hand it back to me by the end of week 9.

Again, no letter grades are actually recorded, but estimated ranges would be C– to C 18-26, B- to B+ 27-37, A- to A 38-50.

All problems were from homework or "proofs to know". References:

1. Hw 6, p. 26, Ex. 1; see p. Y3.

2. (a) Hw 5, p. 73 Ex. 8; (b) Hw 7 p. 95, Ex. 2.

3. (a) Hw 6, Q-7; (b) from proofs to know.

4. From proofs to know.

5. (a) Hw 6, R-2 (see p. Y9); (b) Hw 5, O-3; (c) Hw 4, J-3 and Hw 6, Q-1; (d) Hw 6, p. 21, Ex. 7 and Hw 7, V-9.

Solutions.

1. Method #1: Find the elementary matrices which, when multiplied on the left, change A to RREF. Then multiply the elementary matrices together to get P. [Details omitted]

Method #2: Row-reduce $[A|I] \rightsquigarrow [PA|P]$, as follows:

$$\begin{bmatrix} 1 & 2 & 1 & 5 & 9 & 1 & 0 & 0 \\ 1 & 2 & 2 & 7 & 14 & 0 & 1 & 0 \\ 1 & 2 & 1 & 5 & 9 & 0 & 0 & 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 2 & 1 & 5 & 9 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 & 5 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 \end{bmatrix} \rightsquigarrow$$

$$\begin{bmatrix} 1 & 2 & 0 & 3 & 4 & 2 & -1 & 0 \\ 0 & 0 & 1 & 2 & 5 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 \end{bmatrix}, \text{ so the answer is } P = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}.$$

Check: $PA = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 & 5 & 9 \\ 1 & 2 & 2 & 7 & 14 \\ 1 & 2 & 1 & 5 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 0 & 3 & 4 \\ 0 & 0 & 1 & 2 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$, which is in RREF.

2. (a) Let $T = \tau_A$ with $A = \begin{bmatrix} 2 & 8 & 0 \\ 3 & 1 & 0 \\ 7 & 4 & 0 \end{bmatrix}$.

Reason this is valid: The range of τ_A is the column space of A, which is the span of the first two columns of A since the third column is $\mathbf{0}$.

(b) $\tau_A(v_1) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} = r \begin{bmatrix} 1 \\ 1 \end{bmatrix} + s \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. This gives $r+s=2$, $r-s=1$; solving we get $2r=3$, $r=\frac{3}{2}$, $s=\frac{1}{2}$.

$\tau_A(v_2) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \end{bmatrix} + u \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. This gives $t+u=0$, $t-u=-1$; solving we get $2t=-1$, $t=-\frac{1}{2}$, $u=\frac{1}{2}$.

Putting these results together, we see that the matrix of T with respect to the basis v_1, v_2 is $M = \begin{bmatrix} \frac{3}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$.

3. (a) $v = 5w$, so $w = \frac{1}{5}v$. But which element is $\frac{1}{5}$ (the multiplicative inverse of 5) in $\mathbb{Z}_{31}$? Using the Euclidean algorithm idea, we have

$\begin{bmatrix} 31 & 1 & 0 \\ 5 & 0 & 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 1 & -6 \\ 5 & 0 & 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 1 & -6 \\ 0 & -5 & 31 \end{bmatrix}$. Expressing $(1,1,-6)$ as a linear combination of $(31,1,0)$ and $(5,0,1)$ we get $(1,1,6) = 1\cdot(31,1,0) + (-6)\cdot(5,0,1)$, so $1 = 1\cdot 31 + (-6)\cdot 5$. Working modulo 31, this becomes $1 = 31(-6)$ in $\mathbb{Z}_{31}$. Since $-6 = 25$ in $\mathbb{Z}_{31}$, the answer is that $w = 25v$.

(b) We use the Lemma that a list of vectors $v_1, \ldots, v_n$ is linearly dependent $\Leftrightarrow$ some vector is in the span of the preceding vectors.

Consider w_2, w_1, v_1, v_2, v_3. Since the original list of vectors already spanned V, w_2 is in that span, so this list is linearly dependent. Then by the Lemma, some vector in the list is in the span of the preceding vectors. That vector can't be w_1 or w_2, since they are in a linearly independent set of vectors and so are linearly independent. Therefore it must be some v_i. Then that v_i can be removed without changing the span, which is therefore still V.

4. (a) Since w_1, w_2 are in the range of T, there are v_1, v_2 with $T(v_i) = w_i$ for $i = 1, 2$. The proposed basis is u_1, u_2, u_3, v_1, v_2.

(b) To show that the vectors in (a) are linearly independent, we suppose that a linear combination of them is $\mathbf{0}$ and try to show that the coefficients are all zero. So suppose that for scalars r_1, r_2, r_3, s_1, s_2 we have

(*) $r_1u_1 + r_2u_2 + r_3u_3 + s_1w_1 + s_2w_2 = \mathbf{0}$.

Applying T and using the fact that T is linear, we get

$\mathbf{0} + s_1w_1 + s_2w_2 = \mathbf{0}$ in W. Since w_1, w_2 are a basis of Range(T), they are linearly independent, so $s_1 = s_2 = 0$. Then (*) becomes
$\mathbf{0} + r_1u_1 + r_2u_2 + r_3u_3 = \mathbf{0}$. Since u_1, u_2, u_3 are a basis for the null space of T, they are linearly independent, so $s_1 = s_2 = 0$.

Therefore in (*) all coefficients must be 0, and we have proved the linear independence of the vectors from (a).

5. (a) We know $p(D) = \begin{bmatrix} p(1) & 0 & 0 \\ 0 & p(0) & 0 \\ 0 & 0 & p(-1) \end{bmatrix}$, so we want $p(1) = 0$, $p(0) = 0$, $p(-1) = 0$. In other words, $p(x)$ has roots $1, 0, -1$. Then an answer is $p(x) = (x-1)(x-0)(x+1) = x^3 - x$.

(b) We know that (i) T(subspace) is a subspace and (ii) T^{-1}(subspace) is a subspace. Putting these together, we see that $T^{-1}(T(U))$ is a subspace. It's a subspace of V since we went right and then left.

(Everyone needs more work on what T^{-1} means. If T is invertible then T^{-1} is actually another linear transformation. But whether T is invertible or not, if S is a subspace of W, $T^{-1}(S)$ means the set of all $v \in V$ with $T(v) \in S$, and it's a subspace. This is an application of the notation inf $f(S)$ for subsets.)

(c) This says that for any nonzero vector $v \in \mathbb{R}^2$ there is a $w \in \mathbb{R}_2$ so that vw^t is nonsingular, i.e., has rank 2. This can't be true, since we know vw^t has rank 1. (Recall from homework that vw^t has rank 1 since the rows are all proportional to each other.)

(d) If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ we know that $A^{-1} = \frac{1}{\Delta}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$, where $\Delta = \det A$. Since we are told $\Delta = 1$, the entries of A^{-1} are all plus or minus entries of A, so are integers. Therefore the answer is "yes". (To do this problem, it's not necessary to remember the details of the entries of A^{-1}, just the general idea.)