Solutions to Assignment #7

For p. 95, Ex. 2:

Notice that T is τ_A for $A = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \end{bmatrix}$.

(a) The matrix is A.

(b) We must find coefficients with

$T(\alpha_1) = p\beta_1 + q\beta_2$

$T(\alpha_2) = r\beta_1 + s\beta_2$

$T(\alpha_3) = t\beta_1 + u\beta_2$

Details:

$T(\alpha_1) = A\alpha_1 = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \end{bmatrix} = p \begin{bmatrix} 0 \\ 1 \end{bmatrix} + q \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, so $p = -3$, $q = 1$.

$T(\alpha_2) = A\alpha_2 = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} = r \begin{bmatrix} 0 \\ 1 \end{bmatrix} + s \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, so $r = 1$, $s = 2$.

$T(\alpha_3) = A\alpha_3 = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} = t \begin{bmatrix} 0 \\ 1 \end{bmatrix} + u \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, so $t = -1$, $u = 1$.

Therefore the matrix of T relative to these bases is $\begin{bmatrix} -3 & 1 & -1 \\ 1 & 2 & 1 \end{bmatrix}$.

For p. 95, Ex. 3:

This says $T = \tau_A$. The column space W is the range of T.

For p. 95, Ex. 5.

(Outline) This says $T = \tau_A$. Row-reduce to RREF. The columns of A corresponding to the pivot columns are a basis for the range (the column space). You also know how to find a basis for the null space.

For Problem V-4: (a) You should multiply by the matrix obtained by scaling the rows of I by the same factors, or in other words, by $\begin{bmatrix} 10 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 2 \end{bmatrix}$.

(b) You should multiply on the right by the diagonal matrix from (a).

For Problem V-5: Since P can be obtained by switching the first and third rows of I and also the second and fourth rows, Px is x with the first and third entries switched and also the second and fourth entries.

For Problem V-6: (a) Since N is obtained from the identity matrix by "moving the rows down 1", meaning replacing the first row by 0's and the second row by the first, taking NH produces the same effect on H, so $NH = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ H & E & L & P & I & A & M \end{bmatrix}$.

(b) To make $NH = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ H & E & L & P & I & A & M \end{bmatrix}$ from H we need to "move the rows down 1" by replacing the first row by 0's and second row by the first, so we do that to the identity matrix to deduce that it's N that we want.

(c) N^2 is obtained from N by moving the rows down 1 again, or in other words moving the rows of I down twice, producing the zero matrix.

(d) In the same way, N_4^4 is otained from the identity matrix by moving the rows down four times, so we get the zero matrix.

For Problem V-7:

Let $N = wv^t$. Then $N^2 = (wv^t)^2 = wv^twv^t = w(v^tw)v^t = w0v^t = 0wv^t = 0$ times a 2×2 matrix $=$ the 2×2 zero matrix. In other words, $N^2 = 0$. (Remember, a matrix is nipotent if some power of it is the zero matrix.)

But this derivation may seem a bit mysterious. So consider the example of $(1, 2)$ and $(2, -1)$, which are orthogonal (perpendicular) since their dot product is 0. Write $v = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $w = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$. Then $v^tw = \begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = 0$ (since we can identify 1×1 matrices with scalars). The problem says that wv_t, which is $\begin{bmatrix} 2 \\ -1 \end{bmatrix} \begin{bmatrix} 1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ -1 & -2 \end{bmatrix}$, is nilpotent. In fact, $\begin{bmatrix} 2 & 4 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} 2 & 4 \\ -1 & -2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ so it is nilpotent with only the second power needed to get to the zero matrix.

The solution to the problem is that $(wv^t)^2 = wv^twv^t = w(v^tw)v^t = w0v^t = 0wv^t = 0$ times a 2×2 matrix $=$ the 2×2 zero matrix.

For Problem V-9: We have $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} ad - bc & 0 \\ 0 & ad - bc \end{bmatrix} = \Delta I$, where $\Delta = \det A$.

For Problem V-10:

This problem incorrectly referred to a 3×3 problem. Let's just substitute two examples:

Problem. Using the formula, find the inverse of

(a) $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$; (b) $\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$.

Solutions.

(a) $\Delta = -2$ so the inverse is $\frac{1}{-2}\begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -4 & 2 \\ 3 & -1 \end{bmatrix}$, or $\begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}$.

(b) $\Delta = 2$ so the inverse is $\frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$.

For Problem V-11:

Starting from the *right*, $1 \mapsto 2 \mapsto 3, 2 \mapsto 1 \mapsto 2, 3 \mapsto 3 \mapsto 1$, so the answer is $\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$.

For Problem V-12: $(12)(23) = (123)$ since going right to left, $1 \mapsto 1 \mapsto 2, 2 \mapsto 3 \mapsto 3, 3 \mapsto 2 \mapsto 1$. $(23)(12) = (132)$ since $1 \mapsto 2 \mapsto 3, 3 \mapsto 3 \mapsto 2, 2 \mapsto 1 \mapsto 1$. Therefore they do not commute.

Since the particular symbols do not really matter and since $(ab) = (ba)$, we see that two transpositions that have just one symbol in common do not commute; instead their products are give distinct 3-cycles.

For Problem V-13:

(This problem had a misprint: the last 4 should be 1.)

$1 \mapsto 2 \mapsto 4 \mapsto 7 \mapsto 1$, $3 \mapsto 3$, and $5 \mapsto 6 \mapsto 5$, so the "cycle decomposition" is $(1247)(56)$.

For Problem V-14:

	1	(123)	(132)	(12)	(13)	(23)
1	**1**	(123)	(132)	(12)	(13)	(23)
(123)	(123)	(132)	**1**	(13)	(23)	(12)
(132)	(132)	**1**	(123)	(23)	(12)	(13)
(12)	(12)	(23)	(13)	**1**	(132)	(123)
(13)	(13)	(12)	(23)	(123)	**1**	(132)
(23)	(23)	(13)	(12)	(132)	(123)	**1**

Note: For S_4 the table would be 24×24, and some elements would be non-cycles such as $(12)(34)$.

For Problem V-15: (i) For "$<$": transitive

(ii) For "$\geq$": reflexive, transitive

(iii) For "$=$": all three, so equality is an equivalence relation

(iv) For "$\neq$": symmetric

For Problem V-16:

All of them are equivalence relations:

(v) is reflexive since $x - x = 0$, which is even; symmetric since $x - y$ is even implies $y - x$ is even; transitive since $x - y$ and $y - z$ both even imply $x - z = (x - y) + (y - z)$ is even.

(vi) is reflexive since $f(x) = f(x)$; symmetric since $f(x_1) = f(x_2)$ implies $f(x_2) = f(x_1)$; transitive since if f has equal values at x_1 and x_2 and equal values at x_2 and x_3 then f has equal values at x_1 and x_3.

(vii) is reflexive since $x - x = \mathbf{0} \in W$; symmetric since $x - y \in W$ implies $y - x = -(x - y) \in W$; transitive since $x - y \in W$ and $y - z \in W$ imply $x - z = (x - y) + (y - z) \in W$.

(viii) is reflexive since A has the same row space as itself; symmetric since if A and B have the same row space then B and A have the same row space; transitive since if A and B have the same row space and B and C have the same row space then A and C have the same row space.

(ix) is reflexive since $A = I^{-1}AI$ so $A \sim A$; symmetric since $A \sim B$ means $P^{-1}AP = B$ for some invertible P, and if we multiply on both sides of the equation by P on the left and P^{-1} on the right then we get $A = PBP^{-1}$, which is the same as $A = Q^{-1}BQ$ for $Q = P^{-1}$, giving $B \sim A$; transitive since $A \sim B$ and $B \sim C$ imply $P^{-1}AP = B$ and $Q^{-1}BQ = C$ for some invertible P and Q, so that $Q^{-1}(P^{-1}AP)Q = C$, which is the same as $(PQ)^{-1}A(PQ) = C$, so $A \sim C$.

(x) is all three since any assertion that two elements are related is true, and the three properties consist of such assertions.

(xi) is reflexive since any element is in the same block as itself; symmetric since if x and y are in the same block then y and x are in the same block; transitive since if x and y are in the same block and y and z are in the same block then x and z are in the same block.

For Problem W-1:

Remember, for $F = \mathrm{GF}(q)$, any k-dimensional vector space over F has q elements, even if the vector space is a subspace of some sort. Here we are working inside F^3.

Following the suggestion, for the first row we just need to choose a nonzero vector, so there are q^3-1 possible choices. The first row spans a 1-dimensional subspace of F^3 with q elements.

For the second row, we can choose any triple that avoids this 1-dimensional subspace, so there are $q^3 - q$ possibilities for the second row. The first two rows together are linearly independent, so together they span a 2-dimensional subspace of F^3 with q^2 elements.

For the third row, we can choose any triple that avoids this 2-dimensional subspace, so there are $q^3 - q^2$ possibilities.

Putting these together, we have

$(q^3 - 1)(q^3 - q)(q^3 - q^2) = q^3(q^3 - 1)(q^2 - 1)(q - 1)$

possible matrices. (It is OK to multiply here even though each choice depends on the preceding choices, because the number of possibilities for each choice does *not* depend on the preceding choices.)

For $q = 2$ this gives $2^3 \cdot 7 \cdot 3 \cdot 1 = 168$ invertible matrices.

For Problem W-2:

(a) One way: $e^D = I + \frac{1}{1!}D + \frac{1}{2!}D^2 + \cdots = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \frac{1}{1!}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + \frac{1}{2!}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \cdots = \begin{bmatrix} \sum_{k=0}^{\infty} \frac{1}{k!} & 0 \\ 0 & \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \end{bmatrix} = \begin{bmatrix} e & 0 \\ 0 & e^{-1} \end{bmatrix}$.

Although this is just one example, you can see that the general principle is this: If D is a diagonal matrix with diagonal entries $d_1, \ldots, d_n$ then e^D is a diagonal matrix with diagonal entries $e^{d_1}, \ldots, e^{d_n}$.

Of course, this idea works for other functions in place of e^x. This is another illustration of how the diagonal entries of a diagonal matrix work independently of one another.

(b) Notice that the powers of J are $I, J, -I, -J, I, J, -I, \ldots$. Then $e^{tJ} = I + \frac{1}{1!}tJ + \frac{1}{2!}t^2(-I) + \frac{1}{3!}t^3(-J) + \frac{1}{4!}t^4 I + \ldots$. Collecting terms we get $e^{tJ} = (1 - \frac{1}{2!}t^2 + \frac{1}{4!}t^4 - \ldots)I + (\frac{1}{1!}t - \frac{1}{3!}t^3 + \frac{1}{4!} - \ldots)J = \cos t I + \sin t J = \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix}$, which we recognize as the rotation matrix R_θ with $\theta = t$.

Note. This is a matrix version of Euler's formula about complex numbers: $e^{i\theta} = \cos\theta + i\sin\theta$, the complex number representing a rotation by θ. Putting $\theta = \pi$ gives the famous formula $e^{\pi i} = -1$. You might have seen this in Math 33B. Complex numbers are the subject of Math 132.

(c) This is easier. By the Golden Rule, the powers of N_4 are obtained by successively "sliding the rows down" and its fourth power is the zero matrix.

Therefore $e^{tN_4} = I + \frac{1}{1!}tN_4 + \frac{1}{2!}t^2N_4^2 + \frac{1}{3!}t^3N_4^3 +$ zero matrices, so we get

$$e^{tN_4} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ t & 1 & 0 & 0 \\ \frac{1}{2}t^2 & t & 1 & 0 \\ \frac{1}{6}t^3 & \frac{1}{2}t^2 & t & 1 \end{bmatrix}$$

For Problem W-3: In each case we want to express $T(v_1)$ as a linear combination of v_1 and v_2. Although it is possible to extend the basis and work in $\mathbb{R}^3$, it seems simplest just to stay in V and use unknown coefficients. By the Golden Rule, Pv rotates v downwards one position.

For (a):

$$v_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, T(v_1) = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} = r\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + s\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$v_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, T(v_2) = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = t\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + u\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

This gives three equations in r, s and the same for t, u, so use the two easiest equations in each case. We get $r = -1$, $s = 1$, $t = -1$, $u = 0$. So $M = \begin{bmatrix} r & t \\ s & u \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix}$.

As a check, T is a rotation by $120°$, so $T^3 = \mathbf{1}$. Therefore we should expect $M^3 = I$, which does turn out to be true. (Although we haven't discussed it, M imitates T insofar as such algebraic properties are concerned.)

For (b):

$$v_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, T(v_1) = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} = r\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + s\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$$

$$v_2 = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}, T(v_2) = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} = t\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + u\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$$

This gives three equations in r, s and the same for t, u, so again use the two easiest equations in each case. We get $r = -\frac{1}{2}$, $s = \frac{1}{2}$, $t = -\frac{3}{2}$, $u = -\frac{1}{2}$. So $M = \begin{bmatrix} r & t \\ s & u \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & -\frac{3}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$.

Again, it can be checked that $M^3 = I$.

For Problem W-4:

(a) In the opposite direction, to go from the standard basis to the given nonstandard basis the matrix is $A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$. Then the answer is $M = A^{-1} = \frac{1}{-2}\begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$.

As a check, $\frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

(b) Following the suggestion, A is as in part (a) and $B = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$, giving $M = BA^{-1} = \begin{bmatrix} 2 & -1 \\ 3 & -1 \end{bmatrix}$. Check: $\begin{bmatrix} 2 & -1 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\begin{bmatrix} 2 & -1 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$,

An alternate method: We want M with $M\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $M\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$. We can put these together using the idea that "columns of the matrix on the right work independently" in a matrix product:

$M\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$. This says $MA = B$ with A, B as in the first solution. Now we solve by multiplying both sides on the right by A^{-1}, again getting the solution $M = BA^{-1}$.