Linear systems of differential equations

1. Examples

These systems are "first-order", since they involve only first derivatives. Each one can also be expressed using matrices, if we write $\mathbf{x}$ for $\begin{bmatrix} x \\ y \end{bmatrix}$. Here x and y are functions of t, but we usually write x and y instead of $x(t)$ and $y(t)$.

1.1 *Example.* $\begin{cases} x' = 5x \\ y' = -3y \end{cases}$ with $x(0) = 4$, $y(0) = 2$.

Equivalently, $\mathbf{x}' = D\mathbf{x}$ with $D = \begin{bmatrix} 5 & 0 \\ 0 & -3 \end{bmatrix}$, $\mathbf{x}(0) = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$.

1.2 *Example.* $\begin{cases} x' = x + 2y \\ y' = 2x + y \end{cases}$ with $x(0) = 4$, $y(0) = 2$.

Equivalently, $\mathbf{x}' = A\mathbf{x}$ with $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$, $\mathbf{x}(0) = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$.

1.3 *Example.* $\begin{cases} x' = -2x + 3y \\ y' = -5x + 6y \end{cases}$ with $x(0) = 7$, $y(0) = 3$.

Equivalently, $\mathbf{x}' = A\mathbf{x}$ with $A = \begin{bmatrix} -2 & 3 \\ -5 & 6 \end{bmatrix}$, $\mathbf{x}(0) = \begin{bmatrix} 7 \\ 3 \end{bmatrix}$.

Because "initial data" $\mathbf{x}(0)$ is given, these are "initial value problems".

The system 1.1 can be solved immediately by methods you have had in calculus, where you saw that if x is a function of t and $x' = kx$ then $x = x(0)e^{kt}$. In system 1.1, x and y work independently, so $x = 4e^{5t}$ and $y = 2e^{-3t}$. To write this using matrices, we can say

$$\mathbf{x} = \begin{bmatrix} e^{5t} & 0 \\ 0 & e^{-3t} \end{bmatrix} \begin{bmatrix} 4 \\ 2 \end{bmatrix}.$$

It is not immediately obvious how to solve the nondiagonal systems 1.2 and 1.3. In general the solutions are more complicated, since the rate of change of y affects x and vice-versa.

2. Eigenvectors as initial data

One idea is to look at the case where the **initial data is an eigenvector**. It turns out that in this case you **do** get an **exponential solution**:

2.1 *Proposition.* Suppose A is a 2×2 diagonalizable matrix with eigenvalues λ_1, λ_2 and corresponding linearly independent eigenvectors $\mathbf{v}_1, \mathbf{v}_2$. Then for the linear system $\mathbf{x}' = A\mathbf{x}$,

(i) if $\mathbf{x}(0) = \mathbf{v}_1$, the solution is $e^{\lambda_1 t}\mathbf{v}_1$,

(ii) if $\mathbf{x}(0) = \mathbf{v}_2$, the solution is $e^{\lambda_2 t}\mathbf{v}_2$.

These solutions can be called "principal solutions". Further,

(iii) Any solution is a linear combination of the two principal solutions.

Problem Z-1. (a) Find principal solutions for Example 1.2. (Ignore the initial data given in the example, since the eigenvectors are to be used instead.)

(b) Find principal solutions for Example 1.3, ignoring the initial data given.

Problem Z-2. Prove Proposition 2.1-(i),(ii) together in one calculation, as follows. Suppose λ and $\mathbf{v}$ are an eigenvalue and eigenvector, so that $A\mathbf{v} = \lambda\mathbf{v}$. Let $\mathbf{x} = e^{\lambda t}\mathbf{v}$. Verify that $\mathbf{x}' = A\mathbf{x}$ is true by expanding both sides. In calculating $\mathbf{x}'$, remember that $\mathbf{v}$ is a constant (a constant vector rather than a number, but the familiar rules of calculus still apply). In looking at the right-hand side, remember that a scalar factor can be moved anywhere. Also check that $\mathbf{x}(0)$ is what it is supposed to be.

Problem Z-3. A certain mystery matrix A has characteristic polynomial $\lambda^2 + 5\lambda + 6$. As time goes on, do all nonzero solutions of $\mathbf{x}' = A\mathbf{x}$ tend to increase, do all tend to decrease, or do some solutions increase and others decrease? (Use (iii) of Proposition 2.1.)

From this problem, you can see that you get a lot of information about a linear system just by knowing the eigenvalues, even without finding the eigenvectors!

3. Solution by change of variables

To understand what is happening, we need to make a change of variables. The old variables were x, y, or together $\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$. For new variables let's use r, s, or together $\mathbf{r} = \begin{bmatrix} r \\ s \end{bmatrix}$, as follows. Remember that these variables are all functions of t.

First, given a diagonalizable matrix A, we diagonalize A by finding an invertible matrix P (made from eigenvectors) and a diagonal matrix D (made from eigenvalues) with

3.1 $P^{-1}AP = D.$

Next we define new variables r, s by $\mathbf{r} = P^{-1}\mathbf{x}$, so that

3.2 $\mathbf{x} = P\mathbf{r}.$

Now substitute into the system $\mathbf{x}' = A\mathbf{x}$. We get

3.3 $(P\mathbf{r})' = AP\mathbf{r}$, or since P is constant, $P\mathbf{r}' = AP\mathbf{r}$.

Solve for $\mathbf{r}'$ to get $\mathbf{r}' = P^{-1}AP\mathbf{r}$, or equivalently,

3.4 $\mathbf{r}' = D\mathbf{r}.$

We have diagonalized the linear system of DE's! If D is 2×2, $D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$, and the new system says

3.5 $\left\{ \begin{array}{rcl} r' & = & \lambda_1 r \\ s' & = & \qquad \lambda_2 s \end{array} \right.$

so the solution is

3.6 $\left\{ \begin{array}{rcl} r & = & r(0)e^{\lambda_1 t} \\ s & = & s(0)e^{\lambda_2 t} \end{array} \right.$

To write this solution in matrix form, we should say

3.7 $\mathbf{r} = \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix} \begin{bmatrix} r(0) \\ s(0) \end{bmatrix}.$

A useful shorthand is to agree that any familiar function applied to a diagonal matrix is applied to its diagonal entries. So take D, multiply by t to make $Dt = \begin{bmatrix} \lambda_1 t & 0 \\ 0 & \lambda_2 t \end{bmatrix}$, and then write

3.8 $e^{Dt} = \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix}.$

In this shorthand, then, the solution in terms of the new variables is

3.9 $\mathbf{r} = e^{Dt}\mathbf{r}(0).$

But we need to solve the equation in terms of the original variables. Substitute $\mathbf{r} = P^{-1}\mathbf{x}$ to get $P^{-1}\mathbf{x} = e^{Dt}P^{-1}\mathbf{x}(0)$, so

3.10 $\mathbf{x} = Pe^{Dt}P^{-1}\mathbf{x}(0)$ is the solution to $\mathbf{x}' = A\mathbf{x}$.

Problem Z-4. (a) Solve the system 1.2 by carrying through the steps 3.1 through 3.10 using numbers.

(b) Solve the system 1.3 by doing just step 3.1 and then quoting step 3.10.

In both (a) and (b), expand your answer until you get specific expressions for $x(t)$ and $y(t)$, involving exponentials.

Problem Z-5. In equation 3.9 substitute $\mathbf{r} = P^{-1}\mathbf{x}$ only on the left side and solve for $\mathbf{x}$.

(a) Now put $r(0) = 1, s(0) = 0$, so $\mathbf{r}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, and derive the principal solution 2.1-(i).

(b) Instead, expand the right-hand side and use your knowledge of where P comes from to get

$\mathbf{x} = r(0)e^{\lambda_1 t}\mathbf{v}_1 + s(0)e^{\lambda_2 t}\mathbf{v}_2$. This proves 2.1-(iii).

Problem Z-6. (a) Explain: A square matrix A is singular $\Leftrightarrow$ 0 is an eigenvalue of A. (Method: How does $\det A$ relate to singularity and to eigenvalues?)

(b) Solve the system $\mathbf{x}' = A\mathbf{x}$ for $A = \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix}$, $\mathbf{x}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

4. Other cases

These are for your perspective, but you are not responsible for information in this section on the final examination. The only exception is that higher-order systems may be used in a lab exercise.

- If the eigenvalues of A are complex, the method still works. To take advantage of this possibility, you need to know how complex exponentials relate to trigonometric functions (Euler's formula).

- These ideas also work for $n \times n$ systems $\mathbf{x}' = A\mathbf{x}$ as long as A is diagonalizable, which is usually the case in problems that come from physics.

- There is also a pretty version that works for *any* square matrix A, diagonalizable or not. As you know, the power series for e^x is $e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3 + \cdots$. Likewise, for a square matrix M, define

  $e^M = I + M + \frac{1}{2}M^2 + \frac{1}{3!}M^3 + \cdots$.

  For At in place of M, we get

  $e^{At} = I + At + \frac{1}{2}A^2t^2 + \frac{1}{3!}A^3t^3 + \cdots$.

  It turns out that the solution to $\mathbf{x}' = Ax$ is

  $\mathbf{x} = e^{At}\mathbf{x}(0)$.

- These methods can also be applied to higher-order systems:

  The "order" of a system is the order of the highest derivative that appears. A higher-order linear system can be handled two ways.

1. The first is to substitute new variables to get a larger first-order system. For example, given

   4.1 $$\begin{cases} x'' &= x + 2y \\ y'' &= 2x + y \end{cases}$$

   let $z = x'$ and $w = y'$. Then we get $\begin{cases} x' = z \\ y' = w \\ z' = 2x + y \\ w' = x + 2y \end{cases}$,

   which is a first-order system.

2. For a system such as 4.1 we can write it as is, using matrix notation:

   $\mathbf{x}'' = A\mathbf{x}$. Then the same diagonalization method as in §3 works: substitute new variables, solve a diagonalized system, and substitute back. The only difference is the kind of solution you get:

   $r'' = -4r$, for example, has general solution

   $r = a\cos 2t + b\sin 2t$,

   where $a = r(0)$ and $b = r'(0)$ (two pieces of initial data). Therefore solutions are apt to involve sines and cosines, whose frequencies dependend on the eigenvalues.