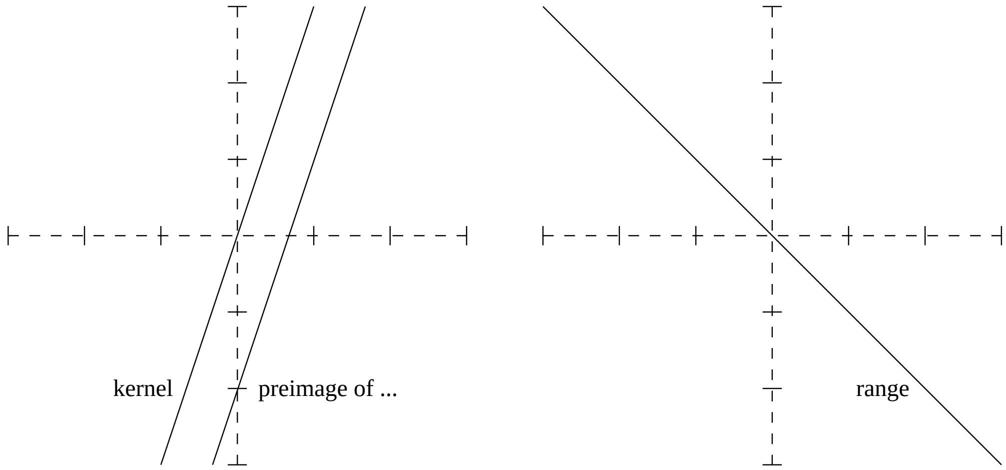
Solutions to Sample Midterm #2

1. Notice that the rows are proportional, so the matrix has rank 1. (a) The range is the column space, which has basis $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and is the line shown. (b) The kernel has basis $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$, so is the line shown. (c) The preimage of $\begin{pmatrix} 2 \\ -2 \end{pmatrix}$ is the set of solutions of $\begin{pmatrix} 3 & -1 \\ -3 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \end{pmatrix}$, which is the same as the solutions of $3x - y = 2$, so $y = 3x - 2$, the line shown. As expected, it is parallel to the kernel.



2. The trace is 7 and the determinant is -8, so the characteristic polynomial is $\lambda^2 - 7\lambda - 8$, which factors as $(\lambda - 8)(\lambda + 1)$, so eigenvalues are 8 and -1. For $\lambda = 8$, $(A - 8I)\mathbf{x} = \mathbf{0}$ says $\begin{pmatrix} -6 & 3 \\ 6 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, so an eigenvector is $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

For $\lambda = -1$, $(A + I)\mathbf{x} = \mathbf{0}$ says $\begin{pmatrix} 3 & 3 \\ 6 & 6 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, so an eigenvector is $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$.

Then $X^{-1}AX = D$ for $X = \begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix}$ and D with diagonal entries $8, -1$. The diagonalization is

$\frac{1}{3}\begin{pmatrix} 1 & 1 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} 2 & 3 \\ 6 & 5 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 8 & 0 \\ 0 & -1 \end{pmatrix}$. This does check upon multiplying out.

3. (a) With X, D as in the preceding problem, multiply the equation $B^3 = A$ on the left by X^{-1} and on the right by X to get $E^3 = D$, where $E = X^{-1}BX$ is unknown. But it's easy to find E: $D = \begin{pmatrix} 8 & 0 \\ 0 & -1 \end{pmatrix}$, so $E = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}$ (the only solution if we don't use complex numbers). Then $B = XEX^{-1} = \begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}\frac{1}{3}\begin{pmatrix} 1 & 1 \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix}$. As a check, BBB does equal A.

4. (a) From homework, the eigenvalues of an $n \times n$ rank 1 matrix $\mathbf{x}\mathbf{y}^T$ are zero with multiplicity $n-1$ and $\mathbf{x} \cdot \mathbf{y}$ (assuming $\mathbf{x} \cdot \mathbf{y} \neq 0$). In the present problem, $n = 3$ and $\mathbf{x} = \mathbf{y} = \mathbf{u}$, a unit vector, so the eigenvalues are $0, 0, 1$.

(b), 1st choice: Let A be the matrix, so $A\mathbf{v}_1 = \lambda_1\mathbf{v}_1$ and $A\mathbf{v}_2 = \lambda_2\mathbf{v}_2$. Suppose $r_1\mathbf{v}_1 + r_2\mathbf{v}_2 = \mathbf{0}$; we must show that $r_1 = r_2 = 0$. Apply L_A to both sides, to get $A(r_1\mathbf{v}_1 + r_2\mathbf{v}_2) = A\mathbf{0}$, which expands to $r_1A\mathbf{v}_1 + r_2A\mathbf{v}_2 = \mathbf{0}$, so $r_1\lambda_1\mathbf{v}_1 + r_2\lambda_2\mathbf{v}_2 = \mathbf{0}$. Also multiply $r_1\mathbf{v}_1 + r_2\mathbf{v}_2 = \mathbf{0}$ on both sides by λ_1, to get $r_1\lambda_1\mathbf{v}_1 + r_2\lambda_2\mathbf{v}_2 = \mathbf{0}$. Subtract the two equations. The first terms cancel and we get $r_2(\lambda_2 - \lambda_1)\mathbf{v}_2 = \mathbf{0}$. Since $\lambda_2 - \lambda_1 \neq 0$ and $\mathbf{v}_2 \neq \mathbf{0}$, the factor r_2 must be 0. Then the original equation becomes $r_1\mathbf{v}_1 + \mathbf{0} = \mathbf{0}$. since $\mathbf{v}_1 \neq \mathbf{0}$, we get $r_1 = 0$. Thus $r_1 = r_2 = 0$, as hoped, and $\mathbf{v}_1, \mathbf{v}_2$ are linearly independent.

5. (a) Since sum of the eigenvalues is the trace and the product is the determinant, the matrix $\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$ (for example) will do.

Another correct approach, used by some people, is to write the characteristic equation $\lambda^2 - \lambda - 1$, set it to zero, and find the roots, which are $\lambda = \frac{1 \pm \sqrt{5}}{2}$, so another answer is the diagonal matrix $\begin{pmatrix} \frac{1+\sqrt{5}}{2} & 0 \\ 0 & \frac{1-\sqrt{5}}{2} \end{pmatrix}$.

(b) (Many people failed to notice that the problem asks for a *diagonal* matrix.) Recall from lecture that a rotation matrix is an orthogonal matrix with determinant $+1$ (rather than -1). A diagonal matrix automatically has mutually orthogonal columns, so an orthogonal matrix is diagonal when the columns have length 1, i.e., the diagonal entries are all ± 1. For a 2×2

matrix, this leaves four possibilities, including I, but only one of the other three has determinant $+1$, namely $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$.

Another way to do this problem is to look at $R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ and consider which values of θ give a diagonal matrix.

(c) Invent A with characteristic polynomial $\lambda^2 - \lambda + 1$, so the trace is 1 and the determinant is 1. An example is $\begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$.

(d) If you're looking for a 2×2 example, to get a 2-dimensional eigenspace you need to have $\lambda_1 = \lambda_2$ and the matrix should not be defective. The only possibility is a scalar matrix such as the zero matrix, the identity matrix, or rI for some other r.

Another way: Choose $\lambda = 0$, so the eigenspace E_λ is the same thing as the nullspace of A. To make a square matrix whose nullspace is two-dimensional, choose a matrix already in rref with two free variables, such as

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$