Sample Midterm #2

[Given Spring '96, based on a different text.]

1. For the linear transformation $L_A : \mathbf{R}^2 \to \mathbf{R}^2$ with $A = \begin{pmatrix} 3 & -1 \\ -3 & 1 \end{pmatrix}$, sketch (a) the image, (b) the kernel, and (c) the inverse image of $\begin{pmatrix} 2 \\ -2 \end{pmatrix}$ [the set of vectors in the domain that go to this vector]. Use two sets of axes. Do a careful job, indicating the scale by "tick marks" on the axes. Label your answers.

2. Diagonalize $A = \begin{pmatrix} 2 & 3 \\ 6 & 5 \end{pmatrix}$, showing the matrices involved, with explicit entries, including finding a matrix inverse where needed. Show that your answer checks, by multiplying out.

3. (a) For $A = \begin{pmatrix} 2 & 3 \\ 6 & 5 \end{pmatrix}$ (as in the problem above), find a matrix B with $B^3 = A$.

[S98: You haven't had a problem like this in homework, but I'll do one in class 5/22 or 5/27.]

4. (a) Find the eigenvalues of the 3×3 matrix $A = \mathbf{u}\mathbf{u}^T$, giving multiplicities, where $\mathbf{u}$ is the unit vector $\begin{pmatrix} \frac{2}{3} \\ \frac{1}{3} \\ \frac{2}{3} \end{pmatrix}$.

[S98: You haven't had a problem like this, but try it anyway by putting together these facts: (i) A matrix of the form $\mathbf{u}\mathbf{v}^t$ (a column vector times a row vector) has rank 1 (as in your homework); (ii) trace(RS) = trace(SR), as you saw in homework; (iii) a row vector times a column vector, where the answer is 1×1, is just a dot product; (iv) an $n \times n$ matrix of rank 1 has just one nonzero eigenvalue; (v) the sum of the eigenvalues of a matrix equals the trace.]

(b) Prove that two eigenvectors $\mathbf{v}_1, \mathbf{v}_2$ belonging to distinct eigenvalues λ_1, λ_2 are linearly independent. [S98: This will be done in class 5/22 or 5/27.]

5. Invent a specific numeric example of each of the following, giving a brief reason in each case:

(a) A 2×2 matrix with eigenvalues whose sum is 1 and whose product is -1.

(b) A 2×2 diagonal matrix that is a rotation but is not the identity matrix.

(c) A matrix A such that $A^2 - A + I = 0$. [S98: Just find a 2×2 matrix A with characteristic polynomial $\lambda^2 - \lambda + 1$. Magically, any matrix acts like a root of its own characteristic polynomial. Check your answer.]

(d) A matrix A and eigenvalue λ so that the eigenspace E_λ has dimension 2. [S98: See the end of Handout T.]