The image/kernel dimension theorem

Background fact: For a linear transformation $T : V \to W$ and $u, v \in V$, $T(u) = T(v) \Leftrightarrow v - u \in \ker T$.

Theorem. Let $T : V \to W$ be a linear transformation between finite-dimensional vector spaces. Then $\dim \ker T + \dim\ \mathrm{im}\ T = \dim V$.

Proof. Let $k = \dim \ker T$ and let $m = \dim\ \mathrm{im}\ T$. We must prove that $k + m = \dim V$.

Choose a basis $v_1, \ldots, v_k$ for $\ker T$ and a basis $w_1, \ldots, w_m$ for $\mathrm{im}\ T$.

For $i = 1, \ldots, m$ choose $u_i \in V$ with $T(u_i) = w_i$.

Let us try to show that $v_1, \ldots, v_k, u_1, \ldots, u_m$ is a basis for V. That will prove the theorem, since then $k + m = \dim V$.

To verify that $v_1, \ldots, v_k, u_1, \ldots, u_m$ are linearly independent, suppose that

(*) $a_1 v_1 + \cdots + a_k v_k + b_1 u_1 + \cdots + b_m u_m = \mathbf{0}$.

We need to show that this linear combination is trivial.

Apply T to (*); since T preserves linear combinations and since the vectors v_i are in the kernel, we get $a_1 \mathbf{0}_1 + \cdots + a_k \mathbf{0}_k + b_1 w_1 + \cdots + b_m w_m = \mathbf{0}$ or more simply, $b_1 w_1 + \cdots + b_m w_m = \mathbf{0}$. Since $w_1, \ldots, w_m$ are linearly independent, $b_1 = \cdots = b_m = \mathbf{0}$. Then from (*) we have $a_1 v_1 + \cdots + a_k v_k = \mathbf{0}$. Since the vectors v_i are linearly independent, we have $a_1 = \cdots = a_k = \mathbf{0}$. Therefore the linear combination in (*) is trivial, as desired.

To show that the vectors $v_1, \ldots, v_k, u_1, \ldots, u_m$ span V, proceed as follows. Give an arbitrary vector $v \in V$, we need to show that there exist scalars $a_1, \ldots, a_k, b_1, \ldots, b_m$ such that

(**) $v = a_1 v_1 + \cdots + a_k v_k + b_1 u_1 + \cdots + b_m u_m$.

Since $w_1, \ldots, w_m$ is a basis for $\mathrm{im}\ T$, we can find $b_1, \ldots, b_m$ with $T(v) = b_1 w_1 + \cdots + b_m w_m$. Then $T(v) = T(b_1 u_1 + \cdots + b_m u_m)$, so the "Background Fact" applies to say that $v - (b_1 u_1 + \cdots + b_m u_m) \in \ker T$. Then $v - (b_1 u_1 + \cdots + b_m u_m) = a_1 v_1 + \cdots + a_k v_k$, for some $a_1, \ldots, a_k$, since the vectors v_i are a basis of $\ker T$. But this is equivalent to $v = a_1 v_1 + \cdots + a_k v_k + b_1 u_1 + \cdots + b_m u_m$, as hoped.

Therefore the vectors $v_1, \ldots, v_k$ and $u_1, \ldots, u_m$ form a basis for V. It follows that $\dim V = k + m = \dim \ker T + \dim\ \mathrm{im}\ T$, as the Theorem states.

Problems.[1]

Problem Q-1. For the case where $T = T_A$ for a matrix A, what does this theorem say, in terms of the rank and "nullity" (dimension of the null space)?

Problem Q-2. Give an example of a vector space V of dimension 4 and vectors $v_1, v_2, u_1, u_2 \in V$ such that v_1, v_2 are linearly independent and u_1, u_2 are linearly independent, but the four together are linearly dependent.

Problem Q-3. Prove this fact: If $T : V \to W$ is a linear transformation between finite-dimensional vector spaces, then it is possible to choose a basis for V and a basis for W such that the matrix of T looks like a matrix of all 0's except possibly for being like an identity matrix in the upper left corner, for example, $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. (Use the image/kernel dimension Theorem.)

[1]To do when assigned.