Solutions/comments for Midterm Exam #1, Version A

Grading.

If you have a question about the grading of your test, please write it on the front of the exam and hand the exam back to me, within the next several days.

The median score was 31. Only scores are recorded, not letter grades, but here are some approximate grade ranges: A– to A 40-50, B– to B+ 31-39, C– to C+ 22-30. People whose grades improve as the course goes on will be given extra consideration at the end, so you can always do better.

Solutions.

1. Since $P(3)$ is a vector space, it is enough to check whether V and W are subspaces of $P(3)$.

For V: Best way: Is the zero function in V? Yes. If you add two polynomials whose value is 0 at $x = 1$, do you get another one? Yes, since $(f + g)(1) = f(1) + g(1) = 0 + 0 = 0$ (reasoning valid for any functions). If you take a polynomial whose value is 0 at $x = 1$ and multiply by a constant, do you get another one? Yes. Then V is a subspace and so is a vector space.

Another way: Write $p(x) = a_0 + a_1x + a_2x^2 + a_3x^3$. Then $p(1) = 0$ says $a_0 + a_1 + a_2 + a_3 = 0$. Does the polynomial with all-zero coefficients fit this description? Yes. If you add two polynomials with this description do you get another one? Yes (explain). If you multiply a polynomial with this description by a constant, do you get another one? Yes. Then V is a subspace and so is a vector space.

For W: Is the zero function in W? No, since its value at $x = 0$ is not 1. So W is not a vector space. OR: Write $p(x) = a_0 + a_1x + a_2x^2 + a_3x^3$. The condition $p(0) = 1$ says $a_0 = 1$ (and the other coefficients can be anything). Does the zero polynomial fit this description? No. Then ...

2. We discussed two ways of finding the basis of the span of a list of vectors.

The first is to put the list of vectors as rows of a matrix, row-reduce to RREF, and take the nonzero rows as the basis. The basis vectors have the I-like pattern of zeros and ones. This fits (a).

The second is to put the list of vectors as columns of a matrix, row-reduce to RREF, and select the *original* vectors corresponding to the pivot columns.

(It's a mistake just to take the pivot columns themselves, since the column space has been changed by row-reduction!) This fits (b).

For (a), you get $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 2 \\ 0 & 2 & 4 \end{bmatrix}$, which row-reduces to $\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$, so the answer is $(1, 0, -2)$ and $(0, 1, 2)$.

For (b), you get the transpose of the matrix in (a), but since it was symmetric it's the same as (a) and the RREF is the same. Then the answer is $\mathbf{u}$ and $\mathbf{v}$.

For (c), any dependencies are the same as for the columns of the RREF in (b), so you can see there is a nontrivial dependency: $\mathbf{w} = -2\mathbf{u} + 2\mathbf{v}$. Therefore $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are linearly dependent.

3. Both were given in lecture and listed on the exam announcement as proofs to know. The reasoning for (a) was this: Since B is obtained from A by an elementary row operation, every row of B is a row of A or at least a linear combination of rows of A, so every row of B is in the row space of A. Then row space$(B) \subseteq$ row space(A). But each elementary row operation is undoable by an elementary row operation, so B can also be obtained from A by an elementary row operation, and so row space$(A) \subseteq$ row space(B). Since the containment is both ways around, the two row spaces are equal.

For (b), the reasoning was given at the end of Handout J. In a problem whose purpose is to work from axioms, it's important to give the reason for each step, by referring to the axioms or to another rule that has been proved from axioms. It's also important not to slip back unconsciously into algebra for numbers, say by writing $\mathbf{u} + \mathbf{v} - \mathbf{w}$ with no parentheses: The axioms for a vector space do not include subtraction between two vectors, just the negative of a single vector, and to omit parentheses means you are unconsciously assuming the associative law.

For model proofs like these, make sure you understand the point of them and ask questions if you don't. If a particular proof seems like a ritual to you instead of a convincing explanation, that means you don't understand it yet.

4. (a) Suppose $r\mathbf{u} + s(\mathbf{u} - \mathbf{v}) + t(\mathbf{u} - \mathbf{v} + \mathbf{w}) = \mathbf{0}$. Expanding and collecting, we get $(r + s + t)\mathbf{u} + (-s - t)\mathbf{v} + t\mathbf{w} = \mathbf{0}$. Since $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are linearly independent, each coefficient must be 0, so that $r + s + t = 0, -s - t = 0, t = 0$. Looking at the third equation, then the second, and then the first, we see that $t = s = r = 0$. Therefore the original three vectors $\mathbf{u}, \mathbf{u} - \mathbf{v}, \mathbf{u} - \mathbf{v} + \mathbf{w}$ are linearly independent.

(b) Using an isomorphism $P(2) \cong \mathbf{R}^3$, the given polynomials correspond to the three vectors of Problem 2, so you can use the work you have already done. Don't forget to write the basis as polynomials! If you use 2(a), you get $1-2x^2, x+2x^2$. If you use 2(b), you get that $1+x$ and $1+2x+2x^2$ are a basis.

5(a) It's not correct as it is, since if $r=s=0$ the equation $r\mathbf{v}+s\mathbf{w}=\mathbf{0}$ is automatically true and proves nothing. The best correction is to say, "If $\mathbf{v}$ and $\mathbf{w}$ are vectors with $r\mathbf{v}+s\mathbf{w}=\mathbf{0}$ and not both of r, s are 0, then $\mathbf{v}$ and $\mathbf{w}$ are linearly dependent." Notice that saying "not both of r, s are 0" is the same as "$r \neq 0$ or $s \neq 0$"; it is *different* from saying $r \neq 0$ *and* $s \neq 0$."

(b) Notice that the vector space here consists of matrices, so the answer should be a list of matrices. This was Example 4-20, and it's also like the homework problem for trace = 0, Ex. 14, p. 216. A good answer is the basis $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$.

(c) As discussed in class, if a vector space has basis $\mathbf{v}_1, \mathbf{v}_2$, then you can write any vector uniquely in the form $a\mathbf{v}_1+b\mathbf{v}_2$, and its coordinate vector is $\begin{bmatrix} a \\ b \end{bmatrix}$. In this problem we need to write $e^{-x}=a\cosh(x)+b\sinh(x)$, which comes out $e^{-x}=1\cdot\cosh(x)-1\cdot\sinh(x)$, so the answer is $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

(d) The simplest answer is to take a matrix that is all zeros except that the first two diagonal entries are 1.