Solutions/comments for sample midterm

(as given to S96 class with their graded exams)

1(a) A basis for the column space of A consists of the first, second, and fifth columns of A. Reason: The same-numbered columns are pivot columns in U and so are a basis for the column space of U; the linear relations are the same for the columns of A as for the columns of U.

1(b) A basis for the row space of A consists of all three rows of U. Reason: All rows of U are pivot rows and so make a basis for the row space of U; the row space of A is the same as the row space of U.

Note. Because the row space of A and U is 3-dimensional, the rows of A are also a basis. Exams with that answer were marked correct, but be careful next time—if only the first two rows of U were nonzero then the first two rows of A might not be a basis for the row space.

1(c) Solve for all variables in terms of free variables and write in terms of column vectors; you get $\begin{pmatrix} x \\ y \\ z \\ s \\ t \end{pmatrix} = z \begin{pmatrix} -2 \\ -3 \\ 1 \\ 0 \\ 0 \end{pmatrix} + s \begin{pmatrix} -3 \\ -4 \\ 0 \\ 1 \\ 0 \end{pmatrix}$, so those two column vectors are a basis for the null space.

2(a) The best way is to check that these polynomials make a subspace of something you already know is a vector space, say of P_3.

Method #1: Polynomials with derivative zero are the constant polynomials. They make a subspace of P_3 because the sum of two constant polynomials is a constant polynomial and multiplying a constant polynomial times a scalar (a constant) gives a constant polynomial.

Method #2 (fancier than you might think of, but you really know it): We discussed how differentiation is a linear transformation. The set of polynomials with $p' = 0$ is the kernel, and the kernel is a subspace.

2(b) The linear relations of the columns of A are the same as the linear relations of the columns of U, so we can answer for columns of U instead:

Column #1 is nonzero. Column #2 is not in the span of column #1. Column #3 is 2 times column #1 plus 3 times column #2. Column #4 is 3 times column #1 plus 4 times column #2. Column #5 is not in the span of the earlier columns.

3(a) The example given in class was $\begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}$, but any matrix whose rank is 1 will do. The image (range) is the span of the columns, so it's a line through the origin in the direction of either column (or if one column is zeros, the nonzero column). The kernel is the null space; just take $\begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ to get $x + y = 0$ (in this example), a line. Draw it. You get a second equation that adds no new information, so discard it (unless the first equation is all zeros).

3(b) The factor is the determinant of your matrix in (a), so is 0 since the rank is 1.

4(a) Method #1: To see if the given polynomials are linearly independent, write a linear combination $= 0$, with unknown coefficients, $a(2 + 2x + x^2) + b(2 + 3x + 3x^2) + c(10 + 13x + 11x^2) = 0$. Collecting coefficients of $1, x, x^2$, you get $(2a+2b+10c)+(2a+3b+13c)x+(a+3b+11c)x^2 = 0$. A polynomial can't equal the zero constant polynomial unless all coefficients are 0, so we know

$$\begin{aligned} 2a + 2b + 10c &= 0 \\ 2a + 3b + 13c &= 0 \\ a + 3b + 11c &= 0 \end{aligned}\ , \text{ or } \begin{pmatrix} 2 & 2 & 10 \\ 2 & 3 & 13 \\ 1 & 3 & 11 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

The coefficient matrix row-reduces to $\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{pmatrix}$, which has a non-pivot column, so there is a nonzero solution for a, b, c. Therefore the original polynomials are linearly dependent, not independent.

Method #2: Realize that polynomials work exactly like the corresponding vectors, so just check whether the vectors $\begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$, $\begin{pmatrix} 2 \\ 3 \\ 3 \end{pmatrix}$, and $\begin{pmatrix} 10 \\ 13 \\ 11 \end{pmatrix}$ are linearly independent. You get the same matrix as in Method #1. (If you are using Method #2 it is good to say what you are doing. Many people just wrote down the matrix and started to work on it.)

Variation: Some people got the matrix and took its determinant. The determinant was zero, so they knew there is a nonzero solution to the equations. That's fine in this problem, but the determinant makes sense only for square matrices. If the problem had listed three polynomials of degree 3 (four coefficients) instead of degree 2, the matrix would have been 4×3 and the determinant method could not be used.

In grading, partial credit was given only if the basic method was correct.

4(b) The proof goes like this: Let S be the span of $\mathbf{v}_1$ and $\mathbf{v}_2$.

(i) To check that S is "closed under addition": Take two typical vectors in S, which you can write as $r_1\mathbf{v}_1+r_2\mathbf{v}_2$ and $s_1\mathbf{v}_1+s_2\mathbf{v}_2$, since they are in the span of $\mathbf{v}_1$ and $\mathbf{v}_2$. Then the sum of these two vectors is $(r_1\mathbf{v}_1+r_2\mathbf{v}_2)+(s_1\mathbf{v}_1+s_2\mathbf{v}_2) = (r_1+s_1)\mathbf{v}_1+(r_1+s_2)\mathbf{v}_2$, which is also a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$ and so is in S.

(ii) To check that S is "closed under multiplication by scalars": Take one typical vector in S, which you can write as $r_1\mathbf{v}_1+r_2\mathbf{v}_2$, and a scalar s. Then $s(r_1\mathbf{v}_1+r_2\mathbf{v}_2) = (sr_1)\mathbf{v}_1+(sr_2)\mathbf{v}_2$, which is also a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$ and so is in S.

Comments. Some people used V instead of S; that doesn't prove anything since all vectors in this problem are already in V. Some people used $L()$, but that's confusing this problem with a proof on linear transformations. Some people wrote $\mathbf{v}_1$ and $\mathbf{v}_2$ when they should have written linear combinations of the two; if you want to give names to the two linear combinations, new names must be used. Some people wrote $(r_1\mathbf{v}_1, r_2\mathbf{v}_2)$ as if it were a pair instead of a sum; that's very different. Some people used a particular vector space such as $\mathbf{R}^n$ or polynomials, but the problem is supposed to be about *any* vector space.

Don't write things like "$S = r_1\mathbf{v}_1+r_2\mathbf{v}_2$", which says a subspace equals a single vector. What you really mean is $S = \{r_1\mathbf{v}_1+r_2\mathbf{v}_2 | r_1, r_2 \in \mathbf{R}\}$, which in words says S is the set of all linear combinations $r_1\mathbf{v}_1+r_2\mathbf{v}_2$ with the coefficients being any real numbers.

For model proofs like these, make sure you understand the point of them and ask questions if you don't. If a particular proof seems like a ritual to you instead of a convincing explanation, that means you don't understand it yet.

5(a) It's not correct as it is, because the part "$r_1\mathbf{v}_1+\cdots+r_k\mathbf{v}_k = \mathbf{0}$ if $r_1 = \cdots = r_k = 0$" is true for any list of vectors, linearly independent or not. Instead of "if" there, you need "implies". In other words, "$r_1\mathbf{v}_1+\cdots+r_k\mathbf{v}_k = \mathbf{0} \Rightarrow r_1 = \cdots = r_k = 0$. It would also be correct to use "if and only if" or "$\Leftrightarrow$".

(b) The kernel of L_A is the same as the null space of A, which you found in $1(c)$.

(c) This is a 3×3 matrix of rank 1, since all the rows are proportional to y, as in a homework problem.

(d) In $\mathbf{R}^3$, the span of four vectors could have any dimension from 0 (in case all four vectors are $\mathbf{0}$) up to 3 (if they span the whole space), but of course not 4.