Comments on homework

Note that solutions to most odd problems are in the back of the text.

In the following, "row-reduce" means to row-reduce to RREF.

Exercises 4-3, p. 214

Ex. 12. W can be described more simply as the set of polynomial functions f of the form $f(x) = a + ax + ax^2 + ax^3$.

(a) (i) The 0 polynomial fits this description with $a = 0$, so $0 \in W$.

(ii) To show W is closed under addition: If $f \in W$ and $g \in W$, then $f(x) = a + ax + ax^2 + ax^3$ and $g(x) = b + bx + bx^2 + bx^3$ for some a and b, so $f(x)+g(x) = (a+b)+(a+b)x+(a+b)x^2+(a+b)x^3$, which is again of the same form, so $f + g \in W$.

(iii) To show W is closed under multiplication by scalars: For $f(x) = a + ax + ax^2 + ax^3$ and any scalar r, $rf(x) = ra + rax + rax^2 + rax^3$, which is again of the same form, so $rf \in W$. Therefore $f \in W$ gives $rf \in W$.

(b) Any two elements of W are proportional to each other, so a basis will consist of any nonzero member, for example $1 + x + x^2 + x^3$.

(c) Since W has a basis consisting of one element, W has dimension 1.

Ex. 14. $Z = \{\begin{bmatrix} a & b \\ c & d \end{bmatrix} : a + d = 0\} = \{\begin{bmatrix} a & b \\ c & -a \end{bmatrix} : a, b, c are any numbers\}$, so every member of Z is uniquely a linear combination of the cases where one letter is 1 and the others are 0, specificially, $\begin{bmatrix} a & b \\ c & -a \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, so these three matrices are a basis and the dimension is 3. (This dimension is what you would expect from the fact that you start with four letters to choose and then put on one condition, namely $a + d = 0$.)

Ex. 18. It's easiest to show that S is a basis in the sense that S spans V and is linearly independent. S spans because we are told that every vector has an expression as a linear combination of vectors of S. S is linearly independent because if $S = \{v_1, \ldots, v_n\}$, we always have the trivial linear relation $0v_1 + \cdots + 0v_n = \mathbf{0}$ and since we are told each vector in V (including $\mathbf{0}$) has a *unique* expression as a linear combination from S there can't be a another, nontrivial linear relation.

Our official definition of a basis was that S is a minimal spanning set. Since we just showed that S spans and is linearly independent, S must also be minimal, since in a linearly independent set no vector can be written as a linear combination of other vectors in the set.

Exercises 4-4, p. 231

Ex. 2(a) Method #1: Take transpose; row-reduce; nonzero rows are basis (but write them as columns again).

Method #2: Row-reduce (without taking transpose); see which columns are pivot columns; the same-numbered columns of the original matrix are a basis for the column space.

In fact, wherever you are asked to find a basis for the span of a list of vectors in $\mathbf{R}^n$, you have this choice—unless you are specifically asked to "thin out" or "select from" the original list of vectors to get a basis for their span, in which case Method #2 is intended.

Ex. 2(b) The rank is the number of pivot entries; it doesn't matter whether you took the transpose before row-reducing.

Ex. 4. There are several ways to do this problem, but I like this one:

If a matrix has more columns than rows, when it is row-reduced there are non-pivot variables, so it is possible to find a linear dependence of the columns. If a matrix has more rows than columns, take its transpose to get a linear dependence; then the rows of the original matrix were linearly dependent.

Although it is possible to give a separate explanation for the case of more rows than columns, that is wasteful; it is better just to mention transposing to take advantage of the case that was already solved.

Ex. 6. Use the isomorphism of $P(2)$ with $\mathbf{R}^3$, so instead of $f(x), g(x), h(x)$ you use $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$, and $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

Using Method #1 above gives the identity matrix, so save all three rows; when you put them back as columns and follow the isomorphism back to $P(2)$ you get the basis $1, x, x^2$.

Using Method #2 above shows that the three original columns were already a basis, so use $f(x), g(x), h(x)$ as the basis.

Warning: If the example had been different, the answers could have been more interesting; for example, you could have a problem where $f(x), h(x)$ are a basis and $g(x)$ is in their span.

Ex. 8. Method #1: Write $\mathbf{u} = x\mathbf{v} + y\mathbf{w}$, make this into three linear equations, row-reduce, and see if there is a solution.

Method #2: Put $\mathbf{v}$, $\mathbf{w}$, and $\mathbf{u}$ as columns of a matrix (so it's 3×3), with $\mathbf{u}$ last since we're hoping to see it as a linear combination of the other two. Row-reduce. Now you can see all the linear relations between the columns more clearly. Is the third column a linear combination of the first two? Then in the original matrix the third column is the same linear combination of the first two.

Ex. 12. Straightforward, as on p. 228 and handout H02.

Ex. 14. Since a subspace is a vector space and it's inside $\mathbf{R}^3$, which has dimension 3, the subspace must have a finite basis and so it has a dimension, 0, 1, 2, or 3. The case 0 means that the basis has no elements and the subspace consists just of the $\mathbf{0}$ vector. The case 1 is a line through the origin (since the span of one nonzero vector looks like that). The case 2 is a plane through the origin. The case 3 is the whole of $\mathbf{R}^3$.

Ex. 16. Here you have to distinguish between the idea of a homogeneous system of linear equations and a possibly nonhomogeneous system (the general case).

This problem says that you can find the general solution to a system of linear equations by finding *one* solution and then adding on any solution of the homogeneous system with the same coefficient matrix. This idea may or may not be familiar to you from the past, but you have seen the same idea in solving a linear differential equation by finding *one* solution and then adding a general solution of the corresponding homogeneous differential equation.

(a) For $X = X' + H$, we have $AX = A(X' + H) = AX' + AH = B + \mathbf{0} = B$.

(b) As suggested, if X is any solution of $AX = B$, then let $H = X - X'$, so we get $AH = A(X - X') = AX - AX' = B - B = 0$. Thus H is a solution of the homogeneous system and also $X = X' + H$.

(c) We know the dimension of the null space of A is $n - r$, so a basis of the null space looks like $H_1, \ldots, H_{n-r}$. In (b), H can be written uniquely in terms of this basis. So $X = X' + H = X' + c_1 H_1 + \cdots + c_{n-r} H_{n-r}$.

Ex. 17. Notice that a "particular" solution X' is easy to find once the matrix has been row-reduced to RREF.

Ex. 18. (a) Since the rows are proportional, the RREF is $\begin{bmatrix} 2 & -1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$ and the rank is 1.

(b) Whenever a matrix A has rank 1, there has to be at least one nonzero row $\mathbf{v}$ and row-reduction is going to consist of zeroing out the other rows using $\mathbf{v}$, so the other rows will be proportional to $\mathbf{v}$. Suppose the rows are $x_1\mathbf{v}, x_2\mathbf{v}, x_3\mathbf{v}$. Make the column vector $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$. Try multiplying out $\mathbf{xv}$ as a matrix product; you can see you get exactly A again.

(c) The method just described works no matter what the size of A is.

Note: A better way to say this is to write all vectors as column vectors, so a row vector is a transpose; the statement is that a matrix A has rank 1 $\Leftrightarrow$ $A = \mathbf{u}\mathbf{v}^t$ for nonzero vectors $\mathbf{u}$ and $\mathbf{v}$. In contrast, if you take $\mathbf{u}^t\mathbf{v}$, that's a row times a column, which is a 1×1 matrix, really just a scalar (the dot product of $\mathbf{u}$ and $\mathbf{v}$).

Exercises 5-1, p. 261.

Ex. 3. The functions will be linear when each component is some linear combination of x and y or x, y, z. You will find that (a), (d), (e) are linear but (b) is messed up by the constant 1 and (c) is messed up by xy.

Ex. 4. Notice that this kind of T is exactly the same as what I called a "uniform scaling" in class.

For (a), verify (i) and (ii) in the definition of a linear transformation.

In (b), the vectors e_1, e_2 are stretched, shrunk, or reversed depending on c.

Ex. 8. M_B is the isomorphism discussed in class. The problem says that if you add two vectors then their coefficient vectors (which are column vectors) add, and if you multiply a vector by a scalar then its coefficient vector is multiplied by the same scalar—pretty clear since there is only one choice for the coefficient vector of the answer in each case.

H02-1. $U = \begin{pmatrix} 1 & 3 & -2 & 0 & 2 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$.

Notice that the given conditions describe each column in terms of the previous columns, or in the case of the first column, as being the zero vector or not. The descriptions are really in terms of linear relations.

(Actually, the first column is not an exception; technically, the span of no vectors is the zero vector–the smallest possible subspace–so saying the first column is not in the span of the previous columns is the same as saying it's nonzero.)

H02-2. No; a column not in the span of previous columns has to be a pivot column, and a column that is in the span of the previous columns can be unique expressed as a linear combination of previous pivot columns.

H02-3. Just do some elementary row operations on U to get another matrix A. Keep going until all entries are nonzero as requested.

H02-4. Yes, it would have to be the same as U, because row operations preserve linear relations and so your friend's matrix would have to fit the same description as in H02-1.

H02-5. The proof is the same as H02-4, except that we are now talking about all possible matrices instead of just your example A.

The proof, then, is this: A matrix in RREF can be uniquely described by saying how columns are or are not linear combinations of the previous columns. The pivot columns are the columns that are not in the span of previous columns; the non-pivot columns can be described as particular linear combinations of preceding pivot columns.

Because these descriptions can be phrased in terms of linear relations, which are not affected by elementary row operations, if two people row-reduce a matrix A and each get a matrix in RREF, both those reduced matrices have the same linear relations and so must be the same matrix.

(Ask yourself: In this theorem, why is it important that we are talking about reduced row-echelon form, instead of just the triangular form where we don't try to clear out entries above the diagonal?)