Study Guide for Midterm #1

Correction: Assignment #4 should say §5.1, not §5.4.

Additional pages to look at: pp. 281-282 on isomorphism.

Background in Mathematics

Logic

implications and different ways to express them

"The following are equivalent"

Kinds of numbers

sets of numbers, smaller to larger: $\mathbf{Z}$, $\mathbf{Q}$, $\mathbf{R}$, $\mathbf{C}$

Sets and functions

membership $s \in S$ (s is in S, or s is an element of S, or s is contained in S)

functions $f : S_1 \to S_2$

- Math 31AB discusses $f : \mathbf{R} \to \mathbf{R}$
- Math 32AB discusses $F : \mathbf{R}^2 \to \mathbf{R}$ and $g : \mathbf{R} \to \mathbf{R}^2$ (for example)
- Math 115A discusses $T : V \to W$ (vector spaces)
- f might be
  - one-to-one
  - onto
- if f is both, is "one-to-one correspondence"

$S_1 \subseteq S_2$ (S_1 is *contained in* S_2, or *is a subset of* S_2)

(Notice that the phrase "contained in" is used to describe both $s \in S$ and $S_1 \subseteq S_2$.)

intersection $S_1 \cap S_2$

The "empty set" has no elements.

- Written $\emptyset$.
- Also called the "null set", but don't confuse with the "null space" of a matrix, which is never empty!

Background in matrices and linear equations

given a system of linear equations

can write it as $A\mathbf{x} = \mathbf{b}$

is it homogeneous or not (the case $\mathbf{b} = \mathbf{0}$)?

is it consistent or not?

Consistent means there is at least one solution.

Homogeneous $\Rightarrow$ consistent! (Why?)

solution by row reduction

Row reduction doesn't change the solutions!

reduced row-echelon form of a matrix (RREF)

(Our text says "row-reduced echelon form", same thing.)

pivot entries

pivot columns, pivot variables

non-pivot columns, non-pivot variables

general solution of a system of linear equations

In RREF, solve for pivot variables in terms of non-pivot variables

finding the inverse of a square matrix A by row-reducing $[A|I]$.

Vector spaces

key examples

$\mathbf{R}^2$, $\mathbf{R}^3$, $\mathbf{R}^n$

a plane through the origin in $\mathbf{R}^3$

More generally, any "subspace" of $\mathbf{R}^n$

possible kinds for $\mathbf{R}^2$, $\mathbf{R}^3$

$P(n)$, the space of polynomials of degree at most n

(What is the dimension?)

The focus is just on adding polynomials, mult by scalars

$M(m \times n)$, the space of $m \times n$ matrices

(What is the dimension?)

Focus is just on adding matrices, mult by scalars.

Forget multiplication of two matrices.

solution space of (linear homogeneous) DE such as $y'' = -y$

$F(\mathbf{R})$, the set of all functions from $\mathbf{R}$ to $\mathbf{R}$

operations on functions are done "pointwise" means $f+g$ at x has value $f(x)+g(x)$, etc.

More generally, $F(S)$, the set of all functions from S to $\mathbf{R}$

defining a vector space using axioms

focus on just + and multiplication by scalars

goal: a vector space should follow all rules for these operations that hold in $\mathbf{R}^n$ (for any n)

key rules (axioms)

used for the definition of a vector space

found many decades ago by mathematicians

other basic rules, such as $0\mathbf{v} = \mathbf{0}$

can be logically derived from the axioms, so weren't included in the list of axioms

most of the time:

We use all rules instinctively without pointing them out.

when asked:

We discuss in detail how some rules follow logically from others.

subspaces

W is a *subspace* of a vector space V if $W \subseteq V$ and

(i) $0 \in W$,

(ii) W is "closed under addition",

(iii) W is "closed under multiplication by scalars"

Examples of subspaces

subspaces of $\mathbf{R}^2$, $\mathbf{R}^3$: what are the possibilities?

$P(n)$ is subspace of $F(\mathbf{R})$

the row space of a matrix (span of the rows)

the column space of a matrix (span of the columns)

the solution space of a system of *homogeneous* linear equations

the null space of a matrix (same as solution space of a system of homogeneous linear equations)

Handy: To prove something is a vector space, show it's a subspace of something you already know is a vector space.

Span and bases

linear combinations

$r_1\mathbf{v}_1 + \cdots + r_k\mathbf{v}_k$

Example in $\mathbf{R}^2$: $3\begin{bmatrix} 2 \\ 1 \end{bmatrix} + 5\begin{bmatrix} 4 \\ -1 \end{bmatrix} + 7\begin{bmatrix} 6 \\ 2 \end{bmatrix}$

span of $\mathbf{v}_1, \ldots, \mathbf{v}_k$

means set of all linear combinations of $\mathbf{v}_1, \ldots, \mathbf{v}_k$

is subspace

in fact, span of $\mathbf{v}_1, \ldots, \mathbf{v}_k$ is the smallest subspace containing $\mathbf{v}_1, \ldots, \mathbf{v}_k$

linear relation between $\mathbf{v}_1, \ldots, \mathbf{v}_k$

is a linear combination equaling $\mathbf{0}$

always have $0\mathbf{v}_1 + \cdots + 0\mathbf{v}_k = \mathbf{0}$, but that's the *trivial* linear relation

nontrivial linear relation—means not all coefficients are 0

linearly dependent (LD) vectors

$\mathbf{v}_1, \ldots, \mathbf{v}_k$ are LD if they have a nontrivial linear relation

Equivalently, a set of vectors is LD if one of them is in the span of the others.

Finding linear relation between columns of a matrix A, with coefficients $x_1, \ldots, x_k$, is same thing as solving $A\mathbf{x} = \mathbf{0}$.

In $\mathbf{R}^3$, any four or more vectors are LD.

More generally, m vectors in $\mathbf{R}^n$ are LD for $m > n$.

Can see this by considering the vectors as columns of a matrix.

linearly independent (LI) vectors

no nontrivial linear relations

linear independence is implication:

$r_1\mathbf{v}_1 + \cdots + r_k\mathbf{v}_k = 0 \Rightarrow r_1 = \cdots = r_k = 0$

to decide LI or LD, given some vectors in $\mathbf{R}^m$,

make a matrix A with these vectors as columns,

take RREF,

see if any nonzero solutions to $A\mathbf{x} = \mathbf{0}$.

basis of a vector space

is a minimal spanning set

Theorem. The following are equivalent for a vector space V and $\mathbf{v}_1, \ldots, \mathbf{v}_n \in V$:

(1) $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are a minimal spanning set,

(2) $\mathbf{v}_1, \ldots, \mathbf{v}_n$ span V and are linearly independent,

(3) $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are a maximal linearly linearly independent set in V,

(4) every vector in V is uniquely expressible as a linear combination of $v_1, \ldots, v_n$

$\mathbf{R}^n$ has standard basis $\mathbf{e}_1, \ldots, \mathbf{e}_n$

think of R^n as n-dimensional graph paper.

Other vector spaces: think of as blank paper, no particular "best" basis. Examples:

plane through origin in $\mathbf{R}^3$

solution space of $y'' = y$

English: "one basis, two bases"

consequences of (4)

coordinates with respect to a basis

$$[\mathbf{v}]_B = \begin{bmatrix} r_1 \\ \cdots \\ r_n \end{bmatrix}$$

gives isomorphism $V \cong \mathbf{R}^n$

Use $\mathbf{v} \leftrightarrow \begin{bmatrix} r_1 \\ \cdots \\ r_n \end{bmatrix}$

Can use to solve basis problems by moving them to $\mathbf{R}^n$

Theorem. Any two bases have same number of vectors.

Proof using

isomorphism

fact that for $n < m$, in $\mathbf{R}^n$ any m vectors are LD.

dimension

So makes sense to define the *dimension* of $\mathbf{V}$ to be the size of a basis.

Wouldn't make sense if there could be more than one size for a basis of $\mathbf{V}$.

working with bases

Inside a finite-dimensional vector space:

Can extend any LI list of vectors to form a basis.

Can thin out any spanning list of vectors to form a basis.

Matrices of any size, $m \times n$

- Use in solving systems of equations
  - consistent, inconsistent
  - general solution
  - homogeneous versus nonhomogeneous
- Spaces related to a matrix
  - row space
  - column space
  - null space
- Row-reduced echelon form (RREF) of A.
  - Use to find basis of row space of A.
  - Use to select columns to make a basis for the column space of A.
  - Use to find basis of null space of A.
  - Use to tell whether list of vectors is LI or LD.
  - Tells linear relations between columns of A.
  - Theorem: RREF is unique.
- Dimensions related to a matrix
  - rank
    - row rank = column rank (since both = number of pivot entries), so say just "rank"
    - How to describe a matrix of rank 1 as $\mathbf{x}\mathbf{y}^T$.
  - "nullity" means dimension of null space [handy word]
  - rank = number of pivot entries
  - nullity = number of non-pivot entries
  - So rank + nullity = n (number of columns)
- What is preserved by row reduction?
  - Row space (and so the rank)
  - (*not* column space!)
  - Null space (and so the nullity)
  - Linear relations between columns
    - and so which columns make a basis for the column space
    - also how to express some columns as linear combinations of others

Linear transformations

Definition: $T : V \to W$ is linear transformation if T is compatible with addition, scalar multiplication

It's fine to have $W = V$, so $T : V \to V$

Examples

- rotation in $\mathbf{R}^2$
- reflection in $\mathbf{R}^2$
- isomorphism $V \cong W$
- differentiation $P(2) \to P(1)$
  - also OK to say $P(2) \to P(2)$, or $P(n) \to P(n)$
- a "matrix transformation" T_A

T_A as linear transformation

- definition
  - for a matrix A, say 2×2, define $T_A : \mathbf{R}^2 \to \mathbf{R}^2$ by $T_A(\mathbf{x}) = A\mathbf{x}$
- how to visualize–e.g., the house pictures
- range (image)
- kernel

Meaning of determinant of A:

- The factor by which T_A changes all areas, if A is 2×2, so $T_A : \mathbf{R}^2 \to \mathbf{R}^2$
- or volumes, if A is 3×3, so $T_A : \mathbf{R}^3 \to \mathbf{R}^3$.
- There is a version for $n \times n$ matrices.

For reference: Axioms for a vector space (p. 181)

	Properties of + alone	
(1)	$\mathbf{v}+\mathbf{w}=\mathbf{w}+\mathbf{v}$	commutative law
(2)	$(\mathbf{v}+\mathbf{w})+\mathbf{u}=\mathbf{v}+(\mathbf{w}+\mathbf{u})$	associative law
(3)	There is an element $\mathbf{0}\in V$ so that for any $\mathbf{v}\in V$ we have $\mathbf{v}+\mathbf{0}=\mathbf{v}$	additive identity element
(4)	For each $\mathbf{v}\in V$ there is an element $-\mathbf{v}\in V$ with $\mathbf{v}+(-\mathbf{v})=\mathbf{0}$	additive inverse
	Compatibility of addition and scalar multiplication	
(5)	$r(\mathbf{v}+\mathbf{w})=r\mathbf{v}+r\mathbf{w}$	
	Connection with operations in $\mathbf{R}$	
(6)	$(r+s)\mathbf{v}=r\mathbf{v}+s\mathbf{v}$	for addition in $\mathbf{R}$
(7)	$(rs)\mathbf{v}=r(s\mathbf{v})$	for multiplication in $\mathbf{R}$
(8)	$1\mathbf{v}=\mathbf{v}$	1 does nothing

The associative law (2) has the benefit that to write $\mathbf{u}+\mathbf{v}+\mathbf{w}$ makes sense even without parentheses. In contrast, consider division of numbers: Does $4/1/2$ mean $(4/1)/2$ or $4/(1/2)$? These have different values, so the parentheses are needed.

Technically, negation $(-\mathbf{v})$ is not the same thing as subtraction $(\mathbf{v}-\mathbf{w})$. In fact, subtraction has not been mentioned, but we can define $\mathbf{v}-\mathbf{w}$ to mean $\mathbf{v}+(-\mathbf{w})$.

Some additional properties of a vector space (p. 188, renumbered)

	scalar 0 and vector 0
(9)	$0\mathbf{v}=0$
(10)	$r\mathbf{0}=\mathbf{0}$
(11)	$r\mathbf{v}=\mathbf{0}\Rightarrow r=0$ or $\mathbf{v}=\mathbf{0}$
	minus in V and minus in $\mathbf{R}$
(12)	$(-r)\mathbf{v}=-(r\mathbf{v})$
	Additive cancellation
(13)	$\mathbf{u}+\mathbf{w}=\mathbf{v}+\mathbf{w}\Rightarrow\mathbf{u}=\mathbf{v}$

Basically, all these rules say you can do what comes naturally. If you take Math 110A or Math 117, you'll see (1)-(4) again; they describe an "additive group".

Sample proofs of these additional properties

In each of these proofs, there is a central idea. If you can invent it or remember it, then the rest of the proof is easy to construct.

I think it's best to prove (13) first, since it depends only on properties of addition.

Proof of (13):

Central idea: Cancellation comes from adding on inverses.

Given $\mathbf{u} + \mathbf{w} = \mathbf{v} + \mathbf{w}$, add on $-\mathbf{w}$ to both sides, getting

$(\mathbf{u} + \mathbf{w}) + (-\mathbf{w}) = (\mathbf{v} + \mathbf{w}) + (-\mathbf{w})$ (and being careful to indicate what is being added first).

By the associative law (2), this is the same as

$\mathbf{u} + (\mathbf{w} + -\mathbf{w}) = \mathbf{v} + (\mathbf{w} + -\mathbf{w})$. By (4), this is the same as

$\mathbf{u} + \mathbf{0} = \mathbf{v} + \mathbf{0}$. By (3), this is the same as $\mathbf{u} = \mathbf{v}$, which is what we needed to prove.

Proof of (9):

Central idea: Use $0 + 0 = 0$ in $\mathbf{R}$.

Given $\mathbf{v}$, since $0 + 0 = 0$ in $\mathbf{R}$, we have $(0 + 0)\mathbf{v} = 0\mathbf{v}$.

Expand the left-hand side using (6) to get $0\mathbf{v} + 0\mathbf{v} = 0\mathbf{v}$.

Now it seems we should be able to cancel, but the cancellation property we proved, (13), requires two pieces on each side. So expand the right-hand side using (3) to get
$0\mathbf{v} + 0\mathbf{v} = 0\mathbf{v} + \mathbf{0}$.

Now notice that in the cancellation property (13), we had the term to be canceled on the right, but right now we have it one the left, so apply the commutative law (1) on the right-hand side to get
$0\mathbf{v} + 0\mathbf{v} = \mathbf{0} + 0\mathbf{v}$.

Now apply (13) to get $0\mathbf{v} = \mathbf{0}$, which is what we were trying to prove.

Comment: As you see, the idea is to suppress your urge to use algebraic rules unconsciously, and instead take baby steps that can each be justified by the axioms.

Later on in the course we applied algebraic rules freely without comment, but in principle we could always go back and prove everything starting from the axioms.