Sample Midterm #1

[Actually given S96. The class was using a different textbook, but the material covered was quite similar to S98. The exam was printed with space to write answers.]

There are five problems, each counting 10 points. For credit, your reasoning must be clear.

1. $A = \begin{pmatrix} 2 & 2 & 10 & 14 & -6 \\ 2 & 3 & 13 & 18 & -7 \\ 1 & 3 & 11 & 15 & -3 \end{pmatrix}$ has rref $U = \begin{pmatrix} 1 & 0 & 2 & 3 & 0 \\ 0 & 1 & 3 & 4 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$.

In each of the following, give both the answer and a brief reason:

(a) Find a basis for the column space of A.

(b) Find a basis for the row space of A.

(c) Find a basis for the null space of A.

2. (a) Do the polynomials $p(x)$ of degree at most 2 with $p'(x) = 0$ form a vector space? Give a brief reason for your answer.

(b) For the matrix A in Problem 1, for each column give a statement showing how it relates (or doesn't) to the previous columns, in a way that could be expressed using linear relations. (For the first column, just say it's nonzero.) Explain how you know.

3. (a) Give an example of a 2×2 matrix A for which neither the kernel nor the image of L_A is all of $\mathbf{R}^2$. Draw a picture. [S98: Our text uses T_A instead of L_A.]

(b) For your choice of A in (a), find the factor by which all areas are multiplied when L_A is performed.

4. In the vector space of polynomials of degree at most 2, determine whether the polynomials $2 + 2x + x^2$, $2 + 3x + 3x^2$, and $10 + 13x + 11x^2$ are linearly independent.

(b) Prove that in a vector space V, the span of vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ is a subspace.

(continued)

5. Brief-answer questions. Except in (a), give a brief reason for your answer.

(a) Is the following definition correct or not? If it is, leave it alone; if it is not, fix it. "In a vector space V, to say that the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$ are *linearly independent* means that $r_1\mathbf{v}_1 + \cdots + r_k\mathbf{v}_k = \mathbf{0}$ if $r_1 = \cdots = r_k = 0$."

(b) For A as in Problem 1, find a basis for the kernel of L_A.

(c) For nonzero column vectors $\mathbf{x}, \mathbf{y}$ in $\mathbf{R}^3$, what values could the rank of $A = \mathbf{x}\mathbf{y}^T$ have? (Give one or more answers, depending on what is possible.)

(d) List all possible dimensions that the span of four vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$ in $\mathbf{R}^3$ could have.