Math 115A **s98 GG.**

Sample final exam

1. For the linear transformation $T_A : \mathbf{R}^6 \to \mathbf{R}^2$ with $A = \begin{pmatrix} 1 & 1 & 2 & 2 & 3 & 3 \\ 4 & 4 & 5 & 5 & 6 & 6 \end{pmatrix}$, find bases for the image and kernel of T_A. (Show your work but proofs are not required. Use a method that could be programmed.)

2. (a) Explain how to find the volume of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$, starting from knowing the volume of a unit sphere, by using a linear transformation. (Say which one and how it is used.)

(b) How many separate pieces does the graph of $x^2 + 4xy + y^2 - z^2 = 1$ have? (Show reasoning. Notice that one factor of the characteristic polynomial of the associated matrix is $\lambda + 1$.)

3. (a) Explain carefully why the polynomials $1, x - 1, x^2 - x - 1$ form a basis for the vector space of all polynomials of degree at most 2.

(b) Explain carefully why the rank plus the nullity of any matrix equals the number of columns. (If your reasons are good ones then the explanation does not have to be long. You may quote without proof any facts you need about rank and spaces associated with a matrix.)

4. For the matrix $A = \begin{pmatrix} -2 & 6 \\ -3 & 7 \end{pmatrix}$, find four matrices B with $B^2 = A$. (If you find one solution B, then $-B$ is another solution, so there are really two pairs to find.)

5. For the system of DE's $\begin{cases} x' = 4x + 3y \\ y' = 3x - 4y \end{cases}$ with symmetric coefficient matrix, show how to diagonalize the system using new variables r, s so that the change of variables is a *rotation*. For your answer, give the new system in terms of r, s and give the change of variables explicitly.

6. Starting from the matrix $A = \begin{pmatrix} 1 & 1 & 6 \\ 1 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$, make an orthogonal matrix B by using the Gram-Schmidt method on the columns of A. (The first two columns are already orthogonal, so that saves some trouble.)

7. Find all true implications between these statements about a 3×3 matrix A, and where an implication is not true, give a counterexample. (No proof needed for correct implications. To avoid confusion, do not even write down any arrows that are not true, unless you put a crossout line through them! You may group equivalent statements.)

(1) A is an invertible matrix.

(2) A is an elementary matrix.

(3) 0 is not an eigenvalue of A.

(4) A is diagonalizable.

(5) T_A is an isomorphism.

8. (a) You have had the theorem that $\mathbf{v}_1, \ldots, \mathbf{v}_n$ in a vector space V are a basis for $V \Leftrightarrow$ every vector $\mathbf{v} \in V$ can be represented uniquely as a linear combination of $\mathbf{v}_1, \ldots, \mathbf{v}_n$. Prove the $\Rightarrow$ direction (only).

(b) Prove that if P is an orthogonal matrix, then $P^{-1} = P^T$. Start from the definition of an orthogonal matrix (the columns are orthonormal).

9. This question asks you to put together several ingredients you have had, in a new way:

For the matrix $A = \begin{pmatrix} 0.5 & 0.4 \\ 0.5 & 0.6 \end{pmatrix}$, A^k for large k is essentially a rank 1 matrix. Explain why. (You will be replacing a number close to 0 by 0 itself.) [S98: Diagonalize A so $A \sim D$ and then think about A^k.]

Briefer questions. (3 points each)

(a) Draw axes, print the first name of your favorite musician, and then print it again after applying to it the transformation $T_A : \mathbf{R}^2 \to \mathbf{R}^2$ for $A = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$.

(b) Explain why the set of all 2×2 matrices with trace $= 0$ is a vector space. You may use the fact that all 2×2 matrices make a vector space, when you add them and multiply them by scalars, "entrywise".

(c) Invent a real matrix A that satisfies $A^2 - A + I = 0$.

(d) Invent an explicit example of a matrix A such that $T_A : \mathbf{R}^2 \to \mathbf{R}^2$ preserves areas but reverses orientation. Choose an example that is not diagonal or triangular.

(e) Give an example of a square matrix that has an eigenvalue of multiplicity > 1 but is not a scalar matrix.

(f) Find the dimension of the space of polynomials $p(x)$ of degree at most 3 such that $p(0) = 0$ and also $p'(0) = 0$. (Show work.)

(g) Sketch the kernel of the linear transformation $T_A : \mathbf{R}^2 \to \mathbf{R}^4$ with $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$.

(h) For the transformation $T_A : \mathbf{R}^2 \to \mathbf{R}^2$ with $A = \begin{pmatrix} 5 & 3 \\ 3 & 5 \end{pmatrix}$, write down the matrix of T_A with respect to the basis $\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ of $\mathbf{R}^2$. (These are eigenvectors.)

(i) Let V be the space of solutions of $y'' = y$ as functions of t. Differentiation $\frac{d}{dt} : V \to V$ is a linear transformation. Find two eigenvalues and corresponding eigenvectors for it. (This is simply a matter of fitting definitions to a different kind of vector space. Keep in mind what "vectors" are in this space. Make sure it's clear what you are doing.)

(j) It is tempting to claim that the rank of a square matrix always equals the number of eigenvalues that are nonzero. Give a specific example to show that this is false. (Choose a nilpotent matrix, perhaps?)