Laboratory Assignment #2: Systems of DE's

1. General

This assignment can be done over the Internet using Netscape with Java enabled. (MS Internet Explorer doesn't seem to work well with these labs, because of incompatibilities between versions of Java.) If the motion looks jumpy, first try the "slow motion" setting in the lab exercises and second try using a faster computer. You have accounts both in the PIC Lab and in the Natural Sciences Learning Center (NSLC) lab in Young Hall South.

The target due date of this assignment is **Wednesday, June 10.** Again, hand it in by emailing your answers to `tst115a1@pic.ucla.edu` in the form of regular email. Also as before, the assignment will graded pass-fail.

For handing in your work, follow instructions with each assignment.

The following exercises are accessible from the Math 115A/1 home page `http://www.math.ucla.edu/~baker/115a.1.98s/` For each, click on the appropriate lab link. After a medium-long wait for files to be downloaded, you should get a bar to click on to start the lab.

2. Linked harmonic oscillators

2.1. Background

In class we have talked about first-order linear systems of ODE's, $\mathbf{x}' = A\mathbf{x}$, especially when A is diagonalizable with real eigenvalues. Each principal solution is of the form $\mathbf{x} = e^{\lambda t}\mathbf{v}$, where λ is an eigenvalue and $\mathbf{v}$ is an eigenvector.

The only trouble is that these exponential solutions are not very exciting. A principal solution with $\lambda < 0$ goes towards zero fast and its motion becomes invisible. A principal solution with $\lambda > 0$ gets large fast and goes off the screen.

It is more interesting to look at second-order systems, $\mathbf{x}'' = A\mathbf{x}$, where A is diagonizable with negative eigenvalues: In this case principal solutions are periodic and more general solutions look crazy.

As a warmup, recall that $x'' = -x$ has general solution $x = a\cos t + b\sin t$, where $a = x(0)$ and $b = x'(0)$. If we use only solutions that have velocity $x'(0) = 0$ at time $t = 0$, then $b = 0$ and so we have just $x = x(0)\cos t$.

Now suppose there is a factor of $k > 0$: The DE $x'' = -kx$ has solutions $x = a\cos\sqrt{k}t + b\sin\sqrt{k}t$, or just $x = a\cos\sqrt{k}t$ if we stick to solutions that are motionless at time $t = 0$. Such a solution is periodic. In one unit of time, the argument of the cosine goes from 0 to $\sqrt{k}$, or from 0 to $\frac{\sqrt{k}}{2\pi}$ periods, so that's the *frequency*. Just remember that the frequency is proportional to the square root of k.

Periodic back-and-forth motion of this kind is called "harmonic oscillation".

For systems $\mathbf{x}'' = A\mathbf{x}$, a solution may involve the sum of functions of two different frequencies, for example $\cos\sqrt{3}t + \cos\sqrt{5}t$. In this case we get a crazy graph that never repeats—it is not periodic at all. (To get a period function the frequencies would have to be "rationally related", meaning that their ratio is a fraction. But $\frac{\sqrt{3}}{\sqrt{5}}$ is irrational.)

A system $\mathbf{x}'' = A\mathbf{x}$ (diagonalizable with negative eigenvalues) is called a system of "coupled harmonic oscillators" (or "linked harmonic oscillators"), unless A is diagonal so the oscillations in different coordinates aren't related to each other. Remember, $\mathbf{x}''$ represents acceleration.

2.2. Select the "double oscillator" lab. The idea here is to get a visual intuition for how eigenvectors help to understand a system. There are several "views" to go through, representing increasing understanding.

For your writeup, just describe some things you notice as you do the steps.

View 1: You see two oscillations, x and y. They look wild and nonperiodic.

View 2: Now you see the acceleration vectors as well. Notice how the acceleration tends to slow down or speed up the motion of the point.

View 3: Same as View 2, except that the axes are made x and y axes so that one point represents the joint motion. But the point still moves crazily.

View 4: The point leaves a trace. Eventually you see there is more structure than you might have thought.

View 5: You see the acceleration vector as well as the position vector. Because $\mathbf{x}'' = A\mathbf{x}$, the two are related by the transformation T_A.

Try stopping the action with "SET", dragging the position to a new location, and restarting with "GO". Now try doing that by having the initial position be an eigenvector (as in Lab #1). Do you get a periodic motion? Try finding an eigenvector in a different direction? Do you get a periodic motion with the same period?

View 6: Now you see the whole picture, with eigenspaces. Start the motion with an initial position that is not on an eigenspace. Can you see how the motion is actually a sum of motions along each of the eigenspaces?

2.3. Select the "multiple oscillator" lab. Now there are four linked oscillators, presented in a different way.

View 1: You see the oscillation.

View 2: You see the matrix A. You can diagonalize A and see P. Then you can show the eigenvectors and with a button select one of them (one of the columns of P).

Then go back to View 1. You will see arrows representing the entries of the eigenvector. Click on "SET", drag the initial position to match the eigenvector, then click on "GO" to see the motion. This time you should get a periodic motion. Now go to View 2 and look at the eigenvalues. Try different ones, going between the views. Can you see how larger eigenvalues give faster motion (since $k = -\lambda$ in the discussion above)? *Note.* An eigenvalue of 0 is not really 0; it just is small enough to print out as 0.

3. Orthogonalization Lab

What does it seem to do?