Notes on derangements

This is a summary of material from lecture. You should be able to supply the reasoning. (Only parts B. and D. are on midterm #2.)

A. Toolbox for exponential generating functions

$$\begin{array}{llllllll}
g(x) & \text{for} & a_0, & a_1, & a_2, & a_3, & \cdots \\
xg(x) & \text{for} & 0, & 1\,a_0, & 2\,a_1, & 3\,a_2, & \cdots \\
g'(x) & \text{for} & a_1, & a_2, & a_3, & a_4, & \cdots \\
xg'(x) & \text{for} & 0, & 1\,a_1, & 2\,a_2, & 3\,a_3, & \cdots
\end{array}$$

B. Concepts for derangements

- A *permutation* of a set S is a function $f : S \to S$ that interchanges the elements of S somehow. In other words, f is *one-to-one* (meaning that two different elements never go to the same element) and *onto* (meaning that every element is the image of some other element).

  If S is a finite set, then one-to-one $\Leftrightarrow$ onto.

  For a set $S = \{a, b, c, d\}$ a typical permutation is $\begin{array}{cccc} a & b & c & d \\ \downarrow & \downarrow & \downarrow & \downarrow \\ c & b & d & a \end{array}$, which we can write as $\begin{pmatrix} a & b & c & d \\ c & b & d & a \end{pmatrix}$ for short.

- A *derangement* is a permutation with no “fixed points”; in other words, even element is taken to a different element.
  Example: $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix}$.

- Write D_n for the number of derangements on n symbols, i.e., on an n-element set. If you write out lists of derangements, you get

n	D_n
0	1
1	0
2	1
3	2
4	9
5	44
$\cdots$	$\cdots$

- D_n obeys the recurrence $D_n = (n-1)(D_{n-1} + D_{n-2})$, for $n \geq 2$.

  See below for the derivation.

  Note. The value $D_0 = 1$ is the one that fits this recurrence relation, going backwards from D_2 and D_1. Its interpretation is that in an empty set, you don't have any fixed point.

C. Finding D_n in closed form

Problem. Find a formula for D_n in terms of n, in closed form.

Difficulty. The recurrence relation for D_n is linear homogeneous, but does not have constant coefficients. Instead, we have

$$\begin{array}{lcl} D_2 & = & 1\,D_1 + 1\,D_0 \\ D_3 & = & 2\,D_2 + 2\,D_1 \\ D_4 & = & 3\,D_3 + 3\,D_2 \\ \cdots & \cdots & \quad\cdots \end{array}$$

Method. Let $g(x)$ be the exponential generating function for the sequence $D_0, D_1, D_2, \ldots$. The coefficients of $1, 2, 3, \ldots$ sound sort of like things in the toolbox, so look in the toolbox carefully for similar relationships, putting D_i for a_i. Notice that, for example,

$D_4 = 3D_3 + 3D_2$ involves exactly the same ingredients as in one column of the toolbox table above, for the rows of $g'(x)$, $xg'(x)$, and $xg(x)$ respectively. The same relationship holds for the same rows in all the other columns of the toolbox table. Even the first column works, since $D_1 = 0$.

Generating functions for sequences add and subtract the same as the sequences themselves, so we now know conclusively that

$g'(x) = xg'(x) + xg(x)$.

Now we can find $g(x)$ by putting y for $g(x)$, simplifying algebraically, separating variables, solving the differential equation, and evaluating the constant of integration. We get

$g(x) = e^{-x}\dfrac{1}{1-x}$.

How can we find an expression for D_n, for D_5? Expand both factors on the right into series, getting

$\frac{D_0}{0!} + \frac{D_1}{1!}x + \frac{D_2}{2!}x^2 + \cdots (\frac{1}{0!} - \frac{1}{1!}x + \frac{1}{2!}x^2 - \cdots)(1 + x + x^2 + x^3 + \cdots)$.

Match coefficients of x^5 on both sides. We get $\frac{D_5}{5!} = \frac{1}{0!}\cdot 1 - \frac{1}{1!}\cdot 1 + \frac{1}{2!}\cdot 1 - \cdots - \frac{1}{5!}\cdot 1$, so

$D_5 = 5!(\frac{1}{0!} - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!})$ (a finite expression, not an infinite series).

In general, then, $D_n = n!(\frac{1}{0!} - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} \cdots \pm \frac{1}{n!})$.

D. Details on the recurrence relation

Question. Given a derangement, how can we make another one, on more symbols?

One way. Start with a derangement, say, $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 4 & 2 \end{pmatrix}$. Pick one of the columns, say the one meaning $3 \to 4$. Add a "detour" through 5: Instead of just having $3 \to 4$, as now, make $3 \to 5$ and $5 \to 4$. Then we have $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 1 & 5 & 2 & 4 \end{pmatrix}$. This procedure is guaranteed to produce a derangement. (Why?)

Another way. Again start with a derangement, say $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$. Pick one of the symbols, say 2. Renumber that symbol and all higher ones, increasing them by 1. This makes $\begin{pmatrix} 1 & 3 & 4 \\ 3 & 4 & 1 \end{pmatrix}$, which is really the same permutation in disguise. Now add $2 \to 5$ and $5 \to 2$. You get $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 5 & 4 & 1 & 2 \end{pmatrix}$. Again, this "add-on" procedure is guaranteed to produce a derangement. (Why?)

Now think about running this process backwards. Given $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 1 & 5 & 2 & 4 \end{pmatrix}$, can you tell how it might have been produced using one of the two methods? Yes: You see $5 \to 4$ but not $4 \to 5$, so it is not the add-on method. Look for what goes to 5: You see $3 \to 5$. Now short-circuit $3 \to 5$ and $5 \to 4$ to make $3 \to 4$ and erase the last column. You have shown that this example can be made by the detour method.

How about $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 5 & 4 & 1 & 2 \end{pmatrix}$? Here you see both $5 \to 2$ and $2 \to 5$, so the "add-on" procedure will produce this permutation.

You can see that the same method will work for any permutation on n symbols, $n \geq 2$: If the last column says $n \to i_n$, look for the column $j_n \to n$ and see if $i_n = j_n$. If so, then the add-on method will work, starting from a permutation on $n-2$ elements. Otherwise, the detour method will work, starting from a permutation on $n-1$ elements. Both procedures involve starting with a choice among $n-1$ symbols, so we get

$D_n = (n-1)D_{n-1} + (n-1)D_{n-2}$.