Review of Exponential Generating Functions

Recall that for a sequence $a_0, a_1, a_2, \ldots$, the corresponding *exponential generating function* is
$g(x) = \sum_{k=0}^{\infty} \frac{a_k}{k!} x^k$, or in other words, $a_0 + a_1 x + \frac{a_2}{2} x^2 + \frac{a_3}{3!} x^3 + \cdots$.

Problem T-1. [1] What is the exponential generating function of $1, 1, 1, 1, \ldots$? (Simplify.)

Problem T-2. What is the exponential generating function of the sequence $P(5,0), P(5,1), \ldots, P(5,5), P(5,6) \cdots$.
(Recall that $P(5,2) = 5 \cdot 4 = \frac{5!}{3!}$, etc., while all terms from $P(5,6)$ on are 0. Simplify your answer to a power by recognizing binomial coefficients.)

Problem T-3. Suppose that $g_1(x)g_2(x) = h(x)$, where $g_1(x)$, $g_2(x)$, and $h(x)$ are exponential generating functions for the sequences $a_0, a_1, \ldots$, $b_0, b_1, \cdots$, and $c_0, c_1, \ldots$ respectively. In other words, suppose that
$(\frac{a_0}{0!} + \frac{a_1}{1!}x + \frac{a_2}{2!}x^2 + \cdots)(\frac{b_0}{0!} + \frac{b_1}{1!}x + \frac{b_2}{2!}x^2 + \cdots) = \frac{c_0}{0!} + \frac{c_1}{1!}x + \frac{c_2}{2!}x^2 + \cdots$.
Express $\frac{c_6}{6!}$ as a sum of products of various fractions $\frac{a_i}{i!}$ and $\frac{b_j}{j!}$.

Problem T-4. Suppose there is a pile of four M&M's, consisting of one each of colors brown, green, orange, red, and yellow. Let a_k be the number of ways of choosing k M&M's from the pile, where the order of choice matters. For example, using `B`,`G`,`O`,`R`,`Y` for colors, one choice of three would be `ORG` and another would be `GRO`.

Find the exponential generating function of $a_0, a_1, \cdots$. (Simplify.)

Problem T-5. Let b_k be the number of ways of choosing k cookies from a pile of four indistinguishable ginger cookies and one walnut cookie, if order matters for the one different cookie. For example, writing `g`,`w` for cookies, three ways of choosing four are `ggwg`, `gggw`, and `gggg`.

(a) What are the values of $b_0, b_1, \ldots$?

(b) Find the exponential generating function of this sequence. (Don't try to simplify.)

[1]These problems are not to be handed in unless assigned

Problem T-6. Think about all the ways of choosing 2 M&M's from the first pile and 4 cookies from the second, if the order of choice matters. For example, two of the ways are `gRgGwg` and `wRggGg`.

(a) Write down five more ways.

(b) Count all the ways as follows:

Step 1: Find the number of ways of choosing 2 M&M's (done in Problem 4 above).

Step 2: Find the number of ways of choosing 4 cookies (done in Problem 5 above).

Step 3: Writing `1`, `2` for the piles, find the number of ways in which two `1`'s and four `2`'s can be distributed among six positions, for example `212212`.

(The answer is a binomial coefficient, but write it using factorials so you can see better how it relates to the two piles.)

Now notice that any way of choosing 2 M&M's from the first pile and 4 cookies from the second corresponds to choosing one of the ways in Step 1, one in Step 2, and one in Step 3. Therefore you should multiply to get the final answer.

Problem T-7. Imagine, but do not write out, all ways of choosing 6 objects from the two piles together, where the order matters. In other words, you are thinking about doing the same calculation as in Problem 6, but for all possibilities $6 = 0 + 6$, $6 = 1 + 5$, $6 = 2 + 4$, etc., rather than just $6 = 2 + 4$ alone.

Let c_6 be the number of ways. Explain why

$c_6 = \frac{6!}{0!6!}60a_0b_6 + \frac{6!}{1!5!}15a_1b_5 + \frac{6!}{2!4!}a_2b_4 + \cdots + \frac{6!}{6!0!}a_6b_0.$

Problem T-8. Show how to rewrite the answer to Problem 7 so that it becomes the same as the answer to Problem 3.

This problem illustrates the general principle you can find c_6, the number of ordered choices of 6 from two piles, by using a product of exponential generating functions. But c_6 is not the actual coefficient of x^6, unlike the case of ordinary generating functions. How is it different? Of course, all this is true for any k in place of 6.

Problem T-9. Use exponential generating functions to find the actual value of c_6 in Problem 7.

Problem T-10. Re-do Problem 5 by using a product of two generating functions. Explain your method.