Mathematical Induction

1.1 The idea

Mathematical induction is a method of proof that is often used to prove that a statement involving an integer n is true for all values of $n = 1, 2, 3, \ldots$ (or often, $n = 0, 1, 2, \ldots$). We can think of proving the statement for a particular value of n as knocking over a domino, as in Figure 1.

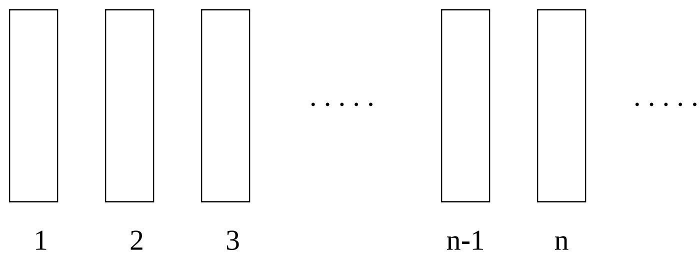

Figure 1: Dominoes

Many kinds of proof proceed by knocking over all dominoes at once, with no special assumptions being used.

In contrast, in mathematical induction we simply show that

- the first domino falls over,
- each domino knocks down the next.

This is often easier, because in proving that each domino falls over, we get to assume something—that the previous domino has fallen over.

The end result is the same: We know that all the dominoes fell down.

1.2 Informal induction

This means that we explain how things work with the first few cases and then leave it to the reader to fill in the rest. In other words, we aren't giving a full proof; we are just giving the idea.

Example. In calculus, suppose we want to prove the power rule $y = x^n \Rightarrow y' = nx^{n-1}$ using the product rule $(uv)' = u'v + uv'$.

This is hard to do for all n at once, because the product rule applies only to a product of two factors, not n factors.

Here is what a proof by informal induction looks like:

$$\begin{array}{llll} y = x & \Rightarrow & y' = 1 & = 1x^0 \\ y = x^2 = x \cdot x & \Rightarrow & y' = 1 \cdot x + x \cdot 1 & = 2x, \\ y = x^3 = x \cdot x^2 & \Rightarrow & y' = 1 \cdot x^2 + x \cdot 2x & = 3x^2, \\ y = x^4 = x \cdot x^3 & \Rightarrow & y' = 1 \cdot x^3 + x \cdot 3x & = 4x^3, \end{array}$$

and so on[1].

Observe how each case (except the first) was proved using the previous one, so that it was not necessary to prove all cases at once from scratch.

1.3 Formal induction

This is just the informal induction written down more carefully.

- Make your statement (or theorem or proposition) involving n:

  Statement. The derivative of x^n is nx^{n-1}

- and say for which n it's supposed to be true: for $n = 1, 2, 3, \ldots$.

- Check the lowest case ($n = 1$ in this example):

  For $n = 1$, the statement says that the derivative of x is $1x^0$, which is true[2].

- Think what the case $n - 1$ says, by taking the statement and putting $n - 1$ in place of n. In our example, the case $n - 1$ says the derivative of x^{n-1} is $(n - 1)x^{n-2}$.

- Show that each domino knocks down the next, or in other words, for unspecified $n > 1$, prove the case n by assuming the case $n - 1$ is true. The reasoning is exactly the same as you would use in the informal induction, except that you just don't say what value n has.

  For $n > 1$, if $y = x^n$, write this as $y = x \cdot x^{n-1}$. By the product rule and the case $n - 1$, we have $y' = 1 \cdot x^{n-1} + x \cdot (n - 1)x^{n-2} = 1 \cdot x^{n-1} + (n - 1) \cdot x^{n-1} = nx^{n-1}$, which proves the case n.

- Finally, say Therefore the statement is true for all n, by induction.

[1]The words "and so on", or "etc.", are often the giveaway that an informal induction is being used.

[2]Technically, at $x = 0$ we need to agree 0^0 means 1.

[2]Or you can go from n to $n + 1$.