Math 113 **F99 DD.**

Solutions to sample final

[These are mostly by Vrej.]

1. (a) $9 + 9 \cdot 9 + 9 \cdot 9 \cdot 8 + 9 \cdot 9 \cdot 8 \cdot 7 = 5274$. (To evaluate, one neat way is $9(1 + 9 + 9 \cdot 8 + 9 \cdot 8 \cdot 7) = \cdots = 9(1 + 9((1 + 8(1 + 7)))) = 9 \cdot (1 + 9 \cdot 65) = 9 \cdot 586 = 5274$.)

(b) 5 with 1 digit, $8 \cdot 5$ with 2 digits, $3 \cdot 1 \cdot 5 + 7 \cdot 8 \cdot 5$ with 3 digits, and $7 \cdot 8 \cdot 1 \cdot 5 + 7 \cdot 1 \cdot 8 \cdot 5 + 6 \cdot 7 \cdot 8 \cdot 5$ with 4 digits, for a total of 2605.

2. The exponential generating function is most appropriate. We want the coefficient of $\frac{x^{10}}{10!}$ in $(\frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots)(1 + x)^{25}$.

3. (a) $a_n = \binom{3n}{3}a_{n-1}$, since you need to choose 3 to put in the last room.

(b) $a_2 = \binom{6}{3}a_1 = 20$.

(c) It is linear and homogeneous but does not have constant coefficients.

(d) Following the suggestion, you get a guess of $a_n = \frac{(3n)!}{(3!)^2}$. Proof by induction: This is true for a_1, since $\frac{3!}{3!} = 1$. Assuming it's true for $n - 1$, looking at the case n and using the recursion we have $a_n = \binom{3n}{3}a_{n-1} = \frac{(3n)!}{3!(3n-3)!}\frac{(3n-3)!}{(3!)^{n-1}} = \frac{(3n)!}{(3!)^n}$, which verifies the case for n. Therefore the assertion is true for all n by induction.

(Having proved the formula, we can now see how it could be derived without the recursion: Line up the people and put them in rooms in order, three at a time. There are $(3n)!$ ways to line them up. Since there are 3! ways to put 3 people in each room, we have overcounted by a factor of $(3!)^n$, so the answer is $\frac{(3n)!}{(3!)^2}$.)

4. (a) We are looking for the coefficient of x^{10} in $(1 + x + x^2 + \cdots)^3$.

(b) We are looking for the coefficient of x^{10} in $(1 + x + \cdots + x^5)^2(x^3 + x^4 + \cdots + x^7)$.

5. (a) This is geometric with ratio x^2, so $F(x) = 1 + x^2 + x^4 + \cdots$.

(b) Reading off from (a), $a_2 = 1$.

(c) Reading off from (a), $\frac{b_2}{2!} = 1$, so $b_2 = 2$.

6. (a) G has a circuit, so it's not a tree.

(b) The answer depends on where you start and what choices you make , but you should get a non-closed chain of length 3.

(Since G is a complete 4-graph, all vertices look the same; more precisely, any permutation of G is an isomorphism of G with itself.)

(c) The spanning tree with edges d, b, e is not produced by the depth-first-search algorithm, since it is not a chain and yet the depth-first-search algorithm produces a chain, as explained in (b).

7. (Omitted.)

8. $e^x = \frac{1}{0!} + \frac{1}{1!}x + \frac{1}{2!}x^2 + \cdots$ so in $e^{2x} = e^x e^x$ the coefficient of x^n is the convolution

$$\frac{1}{0!}\frac{1}{n!} + \frac{1}{1!}\frac{1}{(n-1)!} + \cdots + \frac{1}{n!}\frac{1}{0!} = \sum_{k=0}^{n} \frac{1}{k!(n-k)!}.$$

But also $e^{2x} = \frac{1}{0!} + \frac{1}{1!}2x + \frac{1}{2!}(2x)^2 + \cdots$ so the coefficient of x^n is $\frac{2^n}{n!}$. Putting these two facts together, we get $\displaystyle\sum_{k=0}^{n} \frac{1}{k!(n-k)!} = \frac{2^n}{n!}$, as stated.

(Another way is to notice that multiplying the given equation through by $n!$ it says the sum of the binomial coefficients in one row of Pascal's triangle is 2^n, which is true. So start by mentioning that fact about Pascal's triangle and divide through by $n!$ to get the given equation. However, the intention of the exam was the previous answer.)