The Gamma function

As you know, factorials $n!$ are defined for integer values of n as

$n! = n(n-1)\cdots 1,$

so that the graph of $y = n!$ consists of just a point above each integer n. However, there is a well known function, the **Gamma function** $\Gamma(x)$ that makes it possible to draw a smooth graph through the factorials[1]. The Gamma function is shifted by 1, though, compared to factorials, so we have $\Gamma(n+1) = n!$ when n is an integer. See Figure 1[2].

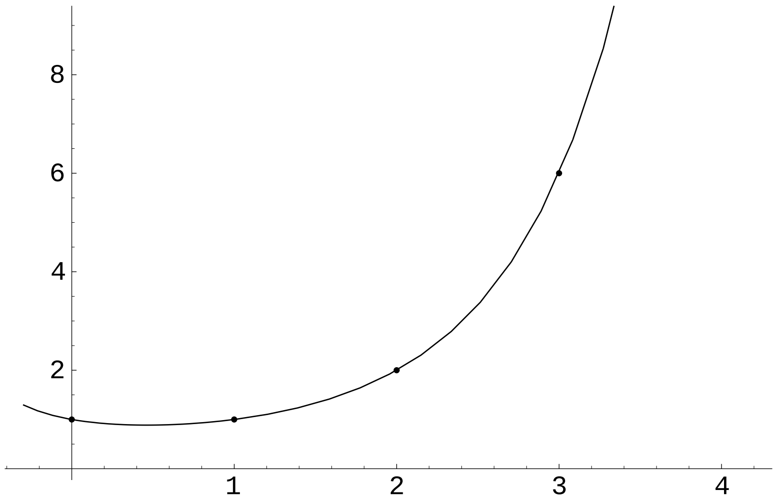

Figure 1: $y = \Gamma(x+1)$

The Gamma function is defined for $x > 0$ by

$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt.$

Notice that x is a constant as far as the integral is concerned. For $x > 1$, as t gets larger the power t^{x-1} is getting bigger and is fighting e^{-t}, which is getting smaller. (Which wins?)

Some people like to turn things around and use the Gamma function to define factorials of non-integer values: Define

$x! = \Gamma(x+1).$

For example, in the Mathematica computer algebra package, if you type `1.5!`, it won't complain; it will simply give you the value of $\Gamma(2.5)$.

[1] The letter Γ is Greek upper-case gamma.
[2] Notice that the two axes in the figure are not drawn to the same scale

Comment:

The procedure just described is analogous to the way we develop a growing understanding of the concept of powers of a positive real number a. We start at the beginning of algebra with the idea of a^n as the product of n copies of a, which makes sense only when n is a positive integer. We later learn that a^x also is defined when x is any real number, so that $y = a^x$ has a smooth graph.

But what does a^x mean, really? There must be some solid definition. The answer starts from a trick that you often use in calculus:

$a^x = (e^{\log a})^x = e^{x \log a} = \exp(x \log a)$. (Often e^x is written as $\exp(x)$.)

So if we can find definitions of $\log(x)$ [meaning base e] and $\exp(x)$ that do not depend on knowing about powers, we can use them to define a^x by saying a^x means $\exp(x \log a)$.

For a good definition for logs, again start with a result you know:

$\int_1^a \frac{1}{x} dx = [\log x]_1^a = \log a - \log 1 = \log a$.

Now turn it into a definition: Define $\log a$ to be $\int_1^a \frac{1}{x} dx$. This is a solid idea since this integral has an existence that doesn't depend on logs.

The exponential function exp can be defined either as the inverse function of log (in other words, $y = \exp(x)$ means $x = \log(y)$) or using a power series. Either way there are some details to check, but everything works fine.

It's interesting that for powers of a there is an integral behind the scenes, just as there is for factorials.

Problem D-1. [3] Check by doing an integral that $\Gamma(1) = 1$, so $\Gamma(1) = 0!$. (Since it's an integral to infinity, you need to integrate from 0 to an unspecified number A and then let A go to infinity.)

Problem D-2. [4] (a) Show that $\Gamma(x+1) = x\Gamma(x)$ for $x > 0$.

You will need to use integration by parts; if you're rusty on that, look it up. Also, for values of x that are less than 1, the value of the "integrand" is undefined for $t = 0$ (so the integral is improper at the low end as well as at the high end). Therefore you need to integrate from h to A and then let $h \to 0$ and $A \to \infty$.

(b) Prove by induction that $\Gamma(n) = (n-1)!$ for $n = 1, 2, \ldots$

[3] Same as C-2, already assigned.

[4] Not to be done until assigned. We'll discuss induction.