

# RESEARCH STATEMENT

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My research focuses on the geometry of spaces with a group action, including the theory of Hamiltonian torus action, equivariant (co)homology theory and its interaction with vertex algebras. In [15] (joint with B. Lian), I generalized the Atiyah and Guillemin-Sternberg convexity theorem (see [1] and [7]) to certain singular spaces (the spaces coming from symplectic reduction), and generalized a theorem of Delzant (see [4]), classifying certain spaces with Hamiltonian torus action by their moment polytopes. In my Ph.D. thesis, fixing the group  $T = \mathbb{C}^*$ , I described the canonical map from the equivariant homology of an orbifold to its equivariant cohomology by a map between the equivariant homology and cohomology of its fixed points. Using this map, I found an approach to calculate the equivariant cohomology ring for  $\overline{M}_{0,n}(\mathbf{G}(m, l), d)$ , which is the coarse moduli space of  $n$ -pointed, genus zero, degree  $d$  stable maps to the Grassmannian  $\mathbf{G}(m, l)$ . People have intensively studied the intersection numbers on  $\overline{M}_{0,n}(\mathbf{G}(m, l), d)$ , known as genus zero Gromov-Witten invariants. I gave a concrete description of the ring  $H_T^*(\overline{M}_{0,0}(\mathbf{G}(m, l), 2))$ .

In [13] and [14] (joint with B. Lian and A. Linshaw), I further developed the chiral equivariant cohomology. The chiral equivariant cohomology, which is defined in [12], is a vertex algebra-valued equivariant cohomology theory for certain differential vertex algebras. Examples of differential vertex algebras in this class are the chiral de Rham complex  $\mathcal{Q}(M)$  of a smooth  $G$ -manifold  $M$ , where  $G$  is a compact Lie group, and the subalgebra  $\mathcal{Q}'(M)$  of  $\mathcal{Q}(M)$  generated by the differential forms  $\Omega(M)$ . I gave some structural theorems for the chiral equivariant cohomology of  $\mathcal{Q}(M)$  and  $\mathcal{Q}'(M)$ , found that for a  $G$ -manifold  $M$ , this theory for  $\mathcal{Q}'(M)$  is strictly stronger than the classical equivariant cohomology theory, and calculated this cohomology algebra for homogeneous space.

## 1 Symplectic reduced space

Let  $G$  be a compact Lie group, and  $M$  be a connected symplectic manifold with a Hamiltonian  $G$ -action and a proper moment map  $J$ . Then the Marsden-Weinstein reduced space  $M_0 = J^{-1}(0)/G$  is a stratified space and has a connected open dense stratum (see [21]). For an exact sequence of compact Lie groups  $1 \rightarrow A \xrightarrow{i} G \rightarrow T \rightarrow 1$ , we assume that  $T$  is a torus and the map  $i^* \circ J$  is proper. The compact reduced space  $X = (i^* \circ J)^{-1}(0)/A$ , which we call *the reduced  $T$ -space*, has a  $T$ -action with a moment map  $J_T$  induced from  $J$ . I proved

**Theorem 1.** *Let  $X$  be a reduced  $T$ -space. Then the induced moment map  $J_T$  has the following properties: 1) the level sets of  $J_T$  are connected; 2) the image of  $J_T$  is convex; 3) the image of  $J_T$  is the convex hull of the image of the fixed points of the  $T$  action.*

A reduced  $T$ -space  $X$  is called a reduced Delzant  $T$ -space if  $\dim X = 2 \dim T$  and  $T$  acts effectively on the dense open stratum of  $X$ . A  $\dim T$  dimensional convex polytope in  $\mathfrak{t}^* = (\text{Lie } T)^*$  is rational if every facet has a normal vector which is rational in  $\mathfrak{t}$ , i.e. it generates a closed subgroup in  $T$ . We have the following correspondence between convex rational polytopes and reduced Delzant  $T$ -spaces.

**Theorem 2.** *The moment polytope of a reduced Delzant  $T$ -space is a convex rational polytope. Every convex rational polytope can be realized as the moment polytope of a reduced Delzant  $T$ -space which has the structure of a complete toric variety. If two reduced Delzant  $T$ -spaces  $(X, J_T)$  and  $(X', J'_T)$  have the same moment polytope then there exists a  $T$ -equivariant homeomorphism  $\phi : X \rightarrow X'$  such that  $J_T = J'_T \circ \phi$ .*

This theorem generalizes results of Delzant in the smooth case (see [4]) and of Lerman-Tolman in the orbifold case (see [10]).

## 2 Equivariant cohomology

Fix  $T = \mathbb{C}^*$ . Let  $E = \varinjlim (\mathbb{C}^m - 0)$ . The equivariant cohomology of a projective  $T$ -space  $X$  can be defined as  $H_T^*(X) = H^*(X \times_T E, \mathbb{C})$ . The equivariant homology is defined to be  $H_n^T(X) = H_n^T(X \times_T (\mathbb{C}^m - 0), \mathbb{C})$  for  $m$  large enough. If  $X$  is an orbifold, the natural map  $H_*^T(X) \rightarrow H_{2 \dim X - *}^T(X)$  given by cap product with the fundamental class is an isomorphism. This map will induce a map of their localization  $\mathbf{A} : H_*(X^T) \otimes \mathbb{C}[t, t^{-1}] \rightarrow H^*(X^T) \otimes \mathbb{C}[t, t^{-1}]$ .  $\mathbf{A}$  can be described concretely using the localization formula.

An approach to find the equivariant cohomology ring is to use the localization theorem. If  $X$  is equivariantly formal, the map  $\iota_X^* : H_T^*(X) \rightarrow H_T^*(X^T)$  is injective. I calculated  $H_T^*(X)$  by two steps: 1) find a base (as  $\mathbb{C}[t]$  module) for  $\iota_X^*(H_T^*(X))$ ; 2) use the product in  $H_T^*(X^T)$  to find the ring structure. Elements of  $\iota_X^*(H_T^*(X))$  can be constructed in two ways:

1. For an equivariant bundle  $V$  on  $X$ , the pullback of the equivariant Chern class is the Chern class of pullback bundle  $\iota_X^* c_i(V) = c_i(\iota_X^* V)$ .
2. Certain equivariant homology classes can be described by their fixed points using the localization formula. Then using  $\mathbf{A}$ , we will get classes in  $\iota_X^*(H_T^*(X))$ .

Using this method, I calculated the equivariant cohomology  $H_T^*(\overline{M}_{0,0}(\mathbf{G}(m, l), 2))$ . Here the action of  $T$  on  $\overline{M}_{0,0}(\mathbf{G}(m, l), 2)$  is induced from the action of  $T$  on  $\mathbf{G}(m, l)$ , which is induced from the action of  $T$  on  $\mathbb{C}^l$  with weights  $-\lambda_1, \dots, -\lambda_l$ . I describe  $H_T^*(\overline{M}_{0,0}(\mathbf{P}^n, 2))$  in the following theorem. The result for  $H_T^*(\overline{M}_{0,0}(\mathbf{G}(m, l), 2))$  is similar.

**Theorem 3.**  $H_T^*(\overline{M}_{0,0}(\mathbf{P}^n, 2))$  is generated by  $t, C_1, C_2, a$ , and the relations are

$$R_n, R_{n+1}, aS_n, aS_{n+1}, a((C_1 - a)^2 - 4C_2)T_{n-1}, a((C_1 - a)^2 - 4C_2)T_n, a((C_1 - a)^2 - 4C_2)T_{n+1}.$$

Let  $\sigma_k, 0 \leq k \leq n+1$  be the elementary symmetric functions of  $\lambda_i, 0 \leq i \leq n$ . Then  $R_k, S_k, T_k$  are defined by

$$\begin{aligned} & \left( \sum_{k=0}^{n+1} \sigma_k t^k x^k \right) \left( \frac{2}{1 + (C_1 - a)x + C_2 x^2} - \frac{ax}{(1 + C_1 x + C_2 x^2)(1 + 1/2 C_1 x)} \right) = \sum_k R_k x^k \\ & 2 \left( \sum_{k=0}^{n+1} \sigma_k t^k x^k \right) \frac{1}{(1 + 1/2 C_1 t)(1 + (1/2 C_1 - a)x)} \\ & + \left( \sum_{k=0}^{n+1} \sigma_k t^k x^k \right) \frac{(1/2(C_1 - a)^2 - 2C_2)x}{(1 + 1/2 C_1 x)(1 + 1/2(C_1 - a)x)(1 + (C_1 - a)x + C_2 x^2)} = \sum_k S_k x^k \\ & \left( \sum_{k=0}^{n+1} \sigma_k t^k x^k \right) \frac{1}{(1 + (C_1 - a)x + C_2 x)(1 + 1/2 C_1 x)} = \sum_k T_k x^k \end{aligned}$$

The ordinary cohomology ring  $H^*(\overline{M}_{0,m}(\mathbf{P}^n, d))$  has been computed in [2] and [19] for  $m = 0$  and in [18] for  $m$  positive by using different methods.

### 3 Chiral equivariant cohomology

Following H. Cartan, Guillemin-Sternberg (see [8]) define the equivariant cohomology  $H_G^*(A)$  for any  $G^*$ -algebra  $A$ . A  $G^*$  algebra is a commutative superalgebra equipped with a  $G$ -action together with a compatible action of the differential Lie superalgebra  $(\mathfrak{sg}, d)$  associated to the Lie algebra  $\mathfrak{g}$  of  $G$ .  $\mathfrak{sg}$  is defined to be  $\mathfrak{g} \triangleright \mathfrak{g}_{-1}$ , where  $\mathfrak{g}_{-1}$  is the adjoint module of  $\mathfrak{g}$  (declared to be an odd vector space), and  $d$  is the derivation of  $\mathfrak{sg}$  which sends  $(x, y) \mapsto (y, 0)$ . Any  $G^*$ -algebra  $A$  is a cochain complex with differential  $d$ . The subalgebra of  $A$  which is both  $G$ -invariant and killed by  $\mathfrak{sg}$  forms a subcomplex  $A_{bas}$  known as the basic subcomplex. The Weil algebra  $W(\mathfrak{g})$  and the differential forms  $\Omega(M)$  on a  $G$ -manifold  $M$  are examples of  $G^*$ -algebras, and the equivariant cohomology  $H_G^*(A)$  of  $A$  is defined to be  $H^*((A \otimes W(\mathfrak{g}))_{bas})$ . In particular,  $H_G^*(\Omega(M))$  equals the equivariant cohomology  $H_G^*(M)$ .

Vertex algebras (see [3], [6], [9] and [16]), arose from conformal field theory, and can be regarded as generalization of commutative algebras. Chiral equivariant cohomology is a vertex algebra analogue of  $H_T^*(A)$ . The analogue of a  $G^*$ -algebra is an  $\mathfrak{sg}[t]$ -algebra, that is, a differential vertex algebra  $\mathcal{A}$  equipped with a linear representation of the Lie subalgebra  $\mathfrak{sg}[t]$  of the loop algebra  $\mathfrak{sg}[t, t^{-1}]$ . The analogue of the Weil algebra  $W(\mathfrak{g})$  is the semi-infinite Weil algebra  $\mathcal{W}(\mathfrak{g})$  (see [5]), which is an  $\mathfrak{sg}[t]$  algebra and contains  $W(\mathfrak{g})$ . The analogues of  $\Omega(M)$  are the chiral de Rham complex  $\mathcal{Q}(M)$  (see [17]), which contains

$\Omega(M)$ , and its subalgebra  $\mathcal{Q}'(M)$  generated by  $\Omega(M)$ . They are both  $\mathfrak{sg}[t]$ -algebras. For any  $\mathfrak{sg}[t]$  algebra  $\mathcal{A}$ ,  $\mathcal{A}_{bas} = \mathcal{A}^{\mathfrak{sg}[t]}$ . The chiral equivariant cohomology  $\mathbb{H}_G^*(\mathcal{A})$  is defined to be  $H^*((\mathcal{A} \otimes \mathcal{W}(\mathfrak{g}))_{bas})$ . In the special cases,  $\mathcal{A} = \mathcal{Q}(M)$  or  $\mathcal{Q}'(M)$ , this equivariant cohomology has a nice structure.

**Theorem 4.**  $\mathbb{H}_G^*(\mathcal{Q}'(M))$  and  $\mathbb{H}_G^*(\mathcal{Q}(M))$  have a quasi conformal structure. If  $G$  is semisimple, then this structure is conformal.

Then  $\mathbb{H}_G^*(\mathcal{Q}'(M))$  and  $\mathbb{H}_G^*(\mathcal{Q}(M))$  have  $\mathbb{Z}_{\geq 0}$  grading by their weights in addition to the degree grading. The weight zero parts of  $\mathbb{H}_G^*(\mathcal{Q}'(M))$  and  $\mathbb{H}_G^*(\mathcal{Q}(M))$  are both equal to the classical equivariant cohomology of  $M$ . When  $M$  is a homogeneous space, the chiral equivariant cohomology is computed by

**Theorem 5.** For any compact, connected group  $G$  and closed subgroup  $H \subset G$ , the positive weight part of  $H_G^*(\mathcal{Q}'(G/H))$  is zero whenever the action is effective up to a finite group. If  $K = \text{Ker}(G \rightarrow \text{Diff}(G/H))$  has positive dimension, then  $\mathbb{H}_G^*(\mathcal{Q}'(G/H)) = \mathbb{H}_{K_0}^*(\mathbb{C}) \otimes H_{G'}^*(G/H)$ . Here  $K_0$  is the identity component of  $K$  and  $G' = G/K_0$ .

So the fixed points of the subgroups generated by ideals of the Lie algebra of  $G$  play a key role in this cohomology. In particular, when  $G$  is simple or  $G$  is the circle  $S^1$ , the positive weight part of  $\mathbb{H}_G^*(M)$  is isomorphic to the positive weight part of  $\mathbb{H}_G^*(M^G)$ . A consequence of this result is  $\mathbb{H}_G^*(\mathcal{Q}')$  is stronger than the classical equivariant cohomology theory: for any simple  $G$ , there exist actions of  $G$  on a disk  $D$  for which  $D^G$  can have any finite homotopy type [20], so the positive weight part of  $\mathbb{H}_G^*(\mathcal{Q}'(M))$  will be different while  $H_G^*(D)$  is always equal to  $S(\mathfrak{g}^*)^G$ . The chiral equivariant cohomology  $\mathbb{H}_G^*(\mathcal{Q}(M))$  is much simpler.

**Theorem 6.** If the action of  $G$  on  $M$  is effective up to a finite group, the positive weight part of  $\mathbb{H}_G^*(\mathcal{Q}(M))$  is zero. If  $K = \text{Ker}(G \rightarrow \text{Diff}(M))$  has positive dimension,

$$\mathbb{H}_G^*(\mathcal{Q}(M)) = \mathbb{H}_{K_0}^* \otimes H_{G'}^*(M),$$

where  $K_0$  denotes the identity component of  $K$  and  $G' = G/K_0$ .

## 4 Future projects

1. I would like to continue to work on the equivariant cohomology of  $\overline{M}_{0,n}(\mathbf{G}(m, l), d)$ . The result I calculated for  $\overline{M}_{0,0}(\mathbf{G}(m, l), 2)$  gives some hint of what this equivariant cohomology ring looks like.
2. The Hilbert scheme  $X^{[n]}$  of points on a smooth projective surface  $X$  is a desingularization of the  $n$ -th symmetric product of  $X$ . The cohomology groups  $H^*(X^{[n]})$  with  $\mathbb{Q}$ -coefficients have relationships with vertex algebras (see [11]). I would like to

compute the equivariant cohomology ring of  $X^{[n]}$  for a projective surface  $X$  equipped with a  $T$ -action. For example if  $X$  is a two dimensional projective toric variety, the fixed points of  $X^{[n]}$  will be easy to determine.

3. I would like to continue working on the chiral equivariant cohomology. A lot of questions arise here. When  $G$  is simple,  $\mathbb{H}_G^*(\mathbb{C})$  is a conformal vertex algebra. This is an essential building block in the theory of chiral equivariant cohomology. The weight zero and weight one parts have been computed. It would be interesting to see what the weight two part is. Let  $\mathbb{H}_G^p(\mathbb{C})[n]$  be the degree  $p$  weight  $n$  part of  $\mathbb{H}_G^*(\mathbb{C})$ . I would like to see what is the generating function

$$\chi(G) = \sum_{p,n} \dim \mathbb{H}_G^p(\mathbb{C})[n] z^p q^n$$

4. Another interesting and important question is the geometric interpretation of the positive weight space of  $\mathbb{H}_G^*(\mathbb{Q}'(M))$ , which may give a bridge connecting this theory to other areas of mathematics and may explain why the positive weight part of  $H^*(\mathbb{Q}'(M))$  is zero while the positive weight part of  $\mathbb{H}_G^*(\mathbb{Q}'(M))$  is not in general.

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