MATH 32B(1) MIDTERM ONE

1. (20 points) Find the area of the part of the surface $z = 4 - x^2 - y^2$, which lies above xy-plane.
answer:

$$D = \{(x,y)|4 - x^2 - y^2 \geq 0\} = \{(r,\theta)|0 \leq r \leq 2, 0 \leq \theta \leq 2\pi\}$$

$$\frac{\partial z}{\partial x} = -2x, \quad \frac{\partial z}{\partial y} = -2y$$

$$Area = \iint_D \sqrt{1 + (\frac{\partial z}{\partial x})^2 + (\frac{\partial z}{\partial y})^2} dA = \iint_D \sqrt{1 + (-2x)^2 + (-2y)^2} dA$$

$$= \int_0^{2\pi} \int_0^2 \sqrt{1 + 4r^2}\, r dr d\theta = \int_0^{2\pi} \frac{1}{12}(1 + 4r^2)^{\frac{3}{2}}|_{r=0}^{r=2} d\theta$$

$$= \int_0^{2\pi} \frac{17^{\frac{3}{2}}}{12} - \frac{1}{12} d\theta = (\frac{17^{\frac{3}{2}}}{6} - \frac{1}{6})\pi$$

2. The joint density function for a pair of random variable X and Y is

$$f(x,y) = C(x^2 + y^2), \quad for\ 0 \leq x \leq 2, 0 \leq y \leq 3$$

and $f(x,y) = 0$ otherwise.

(a) (5 points)Find the value C;

(b) (7 points)Find $P(X \geq 1, Y \leq 1)$;

(c) (8 points) Find $P(X + Y \leq 1)$.

answer: (1)

$$1 = \iint_{\mathbb{R}^2} f(x,y) dA = \int_0^2 \int_0^3 C(x^2 + y^2) dy dx$$

$$\int_0^2 3Cx^2 + 9dx = 26C$$

$$C = \frac{1}{26}$$

(2)

$$P(X \geq 1, Y \leq 1) = \iint_D f(x,y) dA = \int_1^2 \int_0^1 \frac{1}{26}(x^2 + y^2) dy dx$$

$$= \int_1^2 \frac{1}{26}(x^2 + \frac{1}{3}) dx = \frac{4}{39}$$

(3)

$$P(X + Y \leq 1) = \iint_D f(x,y) dA = \int_0^1 \int_0^{1-x} \frac{1}{26}(x^2 + y^2) dy dx$$

$$= \int_0^1 \frac{1}{26}(x^2(1-x) + \frac{1}{3}(1-x)^3) dx = \frac{1}{156}$$

3. (20 points)Find the center of the mass of the lamina bounded by $y = x^2$ and $y = 2 - x^2$ with constant density function.

answer: First find the region

$$x^2 = 2 - x^2, \qquad x = \pm 1$$

$$D = \{(x, y) | -1 \le x \le 1, x^2 \le y \le 2 - x^2\}$$

Assume the density function is $\rho(x, y) = K$. The mass

$$m = \iint_D K dA = \int_{-1}^{1} \int_{x^2}^{2-x^2} K dy dx$$

$$= \int_{-1}^{1} K(2 - 2x^2) dx = \frac{8}{3} K$$

$$\bar{x} = \frac{1}{m} \iint_D xK dA = \frac{1}{m} \int_{-1}^{1} \int_{x^2}^{2-x^2} xK dy dx$$

$$= \frac{1}{m} \int_{-1}^{1} Kx(2 - 2x^2) dx = 0$$

$$\bar{y} = \frac{1}{m} \iint_D yK dA = \frac{1}{m} \int_{-1}^{1} \int_{x^2}^{2-x^2} yK dy dx$$

$$= \frac{1}{m} \int_{-1}^{1} K(2 - 2x^2) dx = \frac{1}{m} \frac{8}{3} K = 1$$

The center of the mass is $(0, 1)$.

4. (20 points)Evaluate

$$\int_0^{\frac{\pi}{2}} \int_0^1 x \sin(xy) dx dy$$

answer:

$$\int_0^{\frac{\pi}{2}} \int_0^1 x \sin(xy) dx dy = \int_0^1 \int_0^{\frac{\pi}{2}} x \sin(xy) dy dx = \int_0^1 - \cos(xy)|_{y=0}^{y=\frac{\pi}{2}} dx$$

$$= \int_0^1 1 - \cos(\frac{\pi}{2} x) dx = 1 - \frac{2}{\pi}$$

5. (20 points)Use polar coordinate to evaluate

$$\iint_D \frac{1}{\sqrt{x^2 + y^2}} dA \qquad D = \{(x, y) | 2x \le x^2 + y^2 \le 4x\}$$

answer:

$$D = \{(x, y) | 2x \le x^2 + y^2 \le 4x\} = \{(r, \theta) | \frac{-\pi}{2} \le \theta \le \frac{\pi}{2}, 2r \cos\theta \le r^2 \le 4r \cos\theta\}$$

$$= \{(r, \theta) | \frac{-\pi}{2} \le \theta \le \frac{\pi}{2}, 2 \cos\theta \le r \le 4 \cos\theta\}$$

$$\iint_D \frac{1}{\sqrt{x^2 + y^2}} dA = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{2\cos\theta}^{4\cos\theta} \frac{1}{r} r dr d\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2 \cos\theta d\theta = 4$$