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HOMEWORK C1

Problem 1. For $n \in \mathbb{N}$, we define the kernel $K_n : \mathbb{R} \to [0, \infty)$ by setting $K_n(x) = c_n(1 - x^2)^n$ for $|x| \leq 1$ and $K_n(x) = 0$ for $|x| > 1$. Here $c_n > 0$ is chosen so that $\int K_n(x) \, dx = 1$.

(a) Show that for each $\delta > 0$, we have $K_n \to 0$ as $n \to \infty$ uniformly on $\mathbb{R} \setminus (-\delta, \delta)$.

(b) Suppose that $f \in C_c(\mathbb{R})$ and $\text{supp } f \subset [0, 1]$. Define $P_n(x) = (K_n * f)(x) = \int K_n(x-u) f(u) \, du$ for $x \in \mathbb{R}$. Show that for $x \in [0, 1]$, the expression $P_n(x)$ is equal to a polynomial in $x$.

(c) Show that we have uniform convergence $P_n \to f$ as $n \to \infty$ on each compact set $M \subset (0, 1)$.

(d) Use the previous considerations to prove Weierstrass’s approximation theorem: If $[a,b] \subset \mathbb{R}$ is a compact interval, then the set of polynomials is dense in $C([a,b])$.

Hint: First prove this for $[a,b] \subset (0, 1)$.

Problem 2. Let $n \geq 2$ and $p \in (1, n)$. For a function $f \in L^p(\mathbb{R}^n)$, we consider the Riesz potential $I(f)(x) = \int_{\mathbb{R}^n} \frac{|f(y)|}{|x-y|^{n-1}} \, dy$, $x \in \mathbb{R}^n$.

(a) Fix $x \in \mathbb{R}^n$ and suppose that for some $R > 0$, we have $f = 0$ on $\mathbb{R}^n \setminus B(x, R)$. Show that then $I(f)(x) \leq C_1 R \cdot Mf(x)$, where $C_1 = C_1(n) > 0$ and $Mf$ denotes the (uncentered) Hardy-Littlewood maximal function of $f$.

Hint: Decompose $B(x, R)$ into dyadic annuli.

(b) Fix $x \in \mathbb{R}^n$. Suppose that for some $R > 0$, we have $f = 0$ on $B(x, R)$. Show that then $I(f)(x) \leq C_2 R^{1-n/p} \|f\|_p$, where $C_2 = C_2(p, n) > 0$.

(c) Show that $\|I(f)\|_{p'} \leq C_3 \|f\|_p$, where $p' = np/(n-p)$ and $C_3 = C_3(p, n) > 0$.

Hint: Split the given function $f$ into two functions as suggested by (a) and (b). Optimize $R$ to find a good pointwise estimate for $I(f)(x)$.

Problem 3. Let $X$ be a complex Hilbert space.

(a) Show that every orthonormal set $A \subset X$ is contained in a maximal orthonormal set $B \subset X$.

(b) Show that if $(x_n)$ is an orthonormal set in $X$, then $x_n \to 0$ in the weak topology on $X$.

Hint: This follows from the more general condition in Problem 4, but derive this from Bessel’s inequality.

(c) Show that if $X$ is separable and infinite-dimensional, then each maximal orthonormal set $A \subset X$ is countably infinite.

Hint: Use the existence of a countably infinite maximal orthonormal set (Hilbert space basis) as discussed in class.
(d) Show that if $X$ is separable and infinite-dimensional, then there exists a linear isomorphism $T : X \to l^2$ that preserves the inner product.

**Problem 4.** Let $X$ be a complex separable Hilbert space with a Hilbert space basis $(x_n)$.

(a) Show that the infinite series $\sum_n \alpha_n x_n$ with coefficients $\alpha_n \in \mathbb{C}$ converges in $X$ if and only if $\sum_n |\alpha_n|^2 < \infty$.

(b) Show that for a sequence $(y_n)$ in $X$, we have $y_n \to y \in X$ in the weak topology on $X$ if and only if

(i) there exists a constant $C$ such that $\|y_n\| \leq C$ for all $n$;

(ii) $\langle y_n, x_k \rangle \to \langle y, x_k \rangle$ as $n \to \infty$ for each $k$.

(c) Let $(y_n)$ be a sequence in $X$. Show that then there exists $y \in X$ such that $y_n \overset{w}{\to} y$ if and only if

(i) there exists a constant $C$ such that $\|y_n\| \leq C$ for all $n$;

(ii) the sequence $(\langle y_n, x \rangle)$ converges for each $x \in X$. 


HOMEWORK C2

Problem 1. Let $X$ be a (complex) Hilbert space.

(a) Show that if $Y$ is another Hilbert space and $T : X \to Y$ is a bounded linear operator, then there exists a bounded linear operator $T^* : Y \to X$ (the adjoint of $T$) such that

$$\langle T(x), y \rangle = \langle x, T^*y \rangle$$

for all $x \in X$ and $y \in Y$.

*Hint: For fixed $y \in Y$, consider the map $x \mapsto \langle T(x), y \rangle$."

(b) Let $M$ be a closed subspace of $X$. Each $x \in X$ can be uniquely represented in the form $x = y + z$ with $y \in M$ and $z \in M^\perp$. Show that $x \mapsto y$ defines a bounded linear operator $P : X \to X$ (the orthogonal projection of $X$ to $M$) satisfying $P^2 = P$ and $P^* = P$. What is the operator norm $\|P\|$?

(c) Let $P : X \to X$ be a bounded linear operator with $P^2 = P$ and $P^* = P$. Show that then there exists a closed linear subspace $M$ of $X$ such that $P$ is the orthogonal projection of $X$ to $M$.

(d) Let $P_N$ denote the space of trigonometric polynomials of degree at most $N$, i.e. the set of all functions of the form

$$f = \sum_{n=-N}^{N} c_n u_n,$$

where $u_n(t) = e^{int}$ and $c_n \in \mathbb{C}$.

Show that $P_N$ is a closed linear subspace of $L^2(\mathbb{T})$ and that $s_N : L^2(\mathbb{T}) \to L^2(\mathbb{T})$ defined as

$$s_N(f) = \sum_{n=-N}^{N} \hat{f}(n) u_n$$

for $f \in L^2(\mathbb{T})$ is the orthogonal projection of $L^2(\mathbb{T})$ to $P_N$.

Problem 2. (a) Let $(c_n)_{n \in \mathbb{Z}}$ be a sequence of complex numbers with $\sum_n |c_n| < \infty$. Show that then $f(t) = \sum_n c_n e^{int}$ converges uniformly for $t \in \mathbb{R}$ and represents a $2\pi$-periodic continuous function $f$ on $\mathbb{R}$ with $\hat{f}(n) = c_n$.

(b) Suppose $f : \mathbb{R} \to \mathbb{C}$ is a $2\pi$-periodic function in $C^1(\mathbb{R})$. Show that then $\hat{f}(n) = O(1/n)$ as $|n| \to \infty$.

(c) Show that a $2\pi$-periodic function $f : \mathbb{R} \to \mathbb{C}$ is $C^\infty$-smooth if and only if for all $k$, we have $f^{(k)}(n) = O(1/|n|^k)$. *Hint: For one of the directions, show that under suitable assumptions, the series in part (a) can be differentiated term-by-term.
Problem 3. Show that there are functions \( f \in C(T) \) such that for the \( N \)-th partial sum \( s_N(f) \) of its Fourier series, we have \( s_N(f)(0) \not\to f(0) \) as \( N \to \infty \).

Hint: Argue by contradiction. Consider the operators \( \Lambda_N : C(T) \to \mathbb{C} \) given by \( \Lambda_N(f) = s_N(f)(0) \). Show that for their operator norms, we have \( \|\Lambda_N\| \geq \|D_N\|_1 \), where \( D_N \) is the Dirichlet kernel.

Problem 4. Let \( \alpha \in \mathbb{R} \) be an irrational number.

(a) Suppose \( f : \mathbb{R} \to \mathbb{C} \) is a continuous function satisfying \( f(t+1) = f(t) \) for \( t \in \mathbb{R} \). Show that then
\[
\frac{1}{N} \sum_{n=1}^{N} f(n\alpha) \xrightarrow{N \to \infty} \int_{0}^{1} f(t) \, dt.
\]

Hint: First show this for certain types of functions \( f \) for which the sum on the left hand side can be computed explicitly.

(b) For \( x \in \mathbb{R} \), let \( \text{frac}(x) \) denote the fractional part of \( x \). Prove Weyl’s equidistribution theorem: For each interval \( [a,b] \subset [0,1] \),
\[
\lim_{N \to \infty} \frac{\# \{ n \in \{1, \ldots, N\} \mid \text{frac}(n\alpha) \in [a,b] \}}{N} = b - a.
\]

(c) Show that \( \{\text{frac}(n\alpha) \mid n \in \mathbb{N}\} \) is dense in \([0,1]\).
Problem 1. For $n \geq 0$ and $t \in \mathbb{R}$, we consider the Dirichlet kernel
\[ D_n(t) = \sum_{k=-n}^{n} e^{int} = \frac{\sin((n+1)t/2)}{\sin(t/2)} \]
and the Fejér kernel
\[ K_n(t) = \frac{1}{n+1} \sum_{k=0}^{n} D_k(t) = \frac{1}{n+1} \left( \frac{\sin((n+1)t/2)}{\sin(t/2)} \right)^2. \]

(a) Show that
(i) $K_n$ is a non-negative $2\pi$-periodic measurable function on $\mathbb{R}$ with $\frac{1}{2\pi} \int_{[-\pi,\pi]} K_n(t) \, dt = 1$;
(ii) for each $\delta > 0$, we have $\lim_{n \to \infty} \int_{[-\pi,\pi]\setminus[-\delta,\delta]} K_n(t) \, dt = 0$.

(b) Show that a sequence $P_n$ of kernels with the properties (i) and (ii) as in part (a) forms an approximate identity on $\mathbb{T}$, in the sense that for each $f \in C(\mathbb{T})$,
\[ (P_n \ast f)(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} P_n(x-t) f(t) \, dt \to f(x) \]
as $n \to \infty$ uniformly for $x \in \mathbb{R}$.

(c) Let $s_nf$ denote the $n$-th partial sum of the Fourier series of a function $f \in L^1(\mathbb{T})$ and consider
\[ \sigma_n f = \frac{1}{n+1} \sum_{k=0}^{n} s_k f. \]
Show that if $f \in C(\mathbb{T})$ then $\|\sigma_n f - f\|_{\infty} \to 0$ as $n \to \infty$.

Problem 2. Let $f(t) = \sum_n c_ne^{int}$ be a trigonometric polynomial. Its (discrete) Hilbert transform is defined by
\[ Hf(t) = -i \sum_{n \leq -1} c_ne^{int} + i \sum_{n \geq 1} c_ne^{int}. \]
One can show that the Hilbert transform $H$ is bounded on $L^p(\mathbb{T})$ for $1 < p < \infty$. More precisely, there exists a constant $C_p \geq 0$ such that
\[ \|Hf\|_p \leq C_p \|f\|_p \]
for each trigonometric polynomial $f$. (This is a rather difficult theorem that can be used without further justification in this problem.)

(a) Show that for $1 < p < \infty$, the Hilbert transform can be extended uniquely to a bounded linear operator $H : L^p(\mathbb{T}) \to L^p(\mathbb{T})$. 
(b) Let $s_n f$ be the $n$-th partial sum of the Fourier series of the trigonometric polynomial $f$. Show that one can represent $s_n f$ as a sum of four terms involving the operator $H$ and two Fourier coefficients of $f$.

*Hint: One of the terms is* \[
\frac{1}{2i} u_n H(fu_{-n}),
\]

*where* $u_{\pm n}(t) = e^{\pm int}$.

(c) Use the previous facts to show that for each $1 < p < \infty$, there exists $C'_p \geq 0$ such that
\[\|s_n f\|_p \leq C'_p \|f\|_p\]
for all $n \geq 0$ and $f \in L^p(\mathbb{T})$. In other words, the operators $s_n$ have uniformly bounded operator norms on $L^p(\mathbb{T})$.

(d) Use part (c) to show that if $f \in L^p(\mathbb{T})$ with $1 < p < \infty$, then the Fourier series of $f$ converges to $f$ in $L^p(\mathbb{T})$, or equivalently,
\[\|s_n f - f\|_p \to 0\]
as $n \to \infty$.

**Problem 3.** A function $f : \mathbb{R}^n \to \mathbb{C}$ is called a *Schwartz function* if it is $C^\infty$-smooth and if all of its partial derivatives $\partial^\alpha f(x)$ tend to 0 as $|x| \to \infty$ faster than any polynomial rate. More precisely, we require that for each multi-index $\alpha$ and each $N \geq 0$, we have
\[\partial^\alpha f(x) = o((1 + |x|)^{-N})\]
as $|x| \to \infty$.

(a) Show that for $f \in C^\infty(\mathbb{R}^n)$, the last condition is equivalent to requiring that for each multi-index $\alpha$ and each $N \geq 0$, there exists a constant $C = C(\alpha, N) \geq 0$ such that
\[(1 + |x|)^N |\partial^\alpha f(x)| \leq C\]
for all $x \in \mathbb{R}^n$.

(b) Show that the function $x \mapsto e^{-|x|^2}$ is a Schwartz function.

(c) Show that if $f$ and $g$ are Schwartz functions on $\mathbb{R}^n$, then $(g * f)(x)$ is defined for each $x$ and $f * g$ is also a Schwartz function.

**Problem 4.** Let $f \in C(\mathbb{T})$ and suppose that $\hat{f}(n) \geq 0$ for each integer $n$. Show that $\sum_n \hat{f}(n) < \infty$.  
