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HOMEWORK C1

Problem 1. For \( n \in \mathbb{N} \), we define the kernel \( K_n : \mathbb{R} \to [0, \infty) \) by setting \( K_n(x) = c_n(1 - x^2)^n \) for \( |x| \leq 1 \) and \( K_n(x) = 0 \) for \( |x| > 1 \). Here \( c_n > 0 \) is chosen so that \( \int K_n(x) \, dx = 1 \).

(a) Show that for each \( \delta > 0 \), we have \( K_n \to 0 \) as \( n \to \infty \) uniformly on \( \mathbb{R} \setminus (-\delta, \delta) \).

(b) Suppose that \( f \in C_c(\mathbb{R}) \) and \( \text{supp} f \subset [0, 1] \). Define \( P_n(x) = (K_n * f)(x) = \int K_n(x-u)f(u) \, du \) for \( x \in \mathbb{R} \). Show that for \( x \in [0, 1] \), the expression \( P_n(x) \) is equal to a polynomial in \( x \).

(c) Show that we have uniform convergence \( P_n \to f \) as \( n \to \infty \) on each compact set \( M \subset (0, 1) \).

(d) Use the previous considerations to prove Weierstrass’s approximation theorem: If \( [a,b] \subset \mathbb{R} \) is a compact interval, then the set of polynomials is dense in \( C([a,b]) \).

Hint: First prove this for \( [a,b] \subset (0, 1) \).

Problem 2. Let \( n \geq 2 \) and \( p \in (1, n) \). For a function \( f \in L^p(\mathbb{R}^n) \), we consider the Riesz potential

\[
I(f)(x) = \int_{\mathbb{R}^n} \frac{|f(y)|}{|x-y|^{n-1}} \, dy, \quad x \in \mathbb{R}^n,
\]

where \( dy \) indicates integration with respect to Lebesgue measure.

(a) Fix \( x \in \mathbb{R}^n \) and suppose that for some \( R > 0 \), we have \( f = 0 \) on \( \mathbb{R}^n \setminus B(x, R) \). Show that then \( I(f)(x) \leq C_1 R \cdot Mf(x) \), where \( C_1 = C_1(n) > 0 \) and \( Mf \) denotes the (uncentered) Hardy-Littlewood maximal function of \( f \).

Hint: Decompose \( B(x, R) \) into dyadic annuli.

(b) Fix \( x \in \mathbb{R}^n \). Suppose that for some \( R > 0 \), we have \( f = 0 \) on \( B(x, R) \). Show that then \( I(f)(x) \leq C_2 R^{1-n/p} \|f\|_p \), where \( C_2 = C_2(p, n) > 0 \).

(c) Show that \( \|I(f)\|_{p^*} \leq C_3 \|f\|_p \), where \( p^* = np/(n-p) \) and \( C_3 = C_3(p, n) > 0 \).

Hint: Split the given function \( f \) into two functions as suggested by (a) and (b). Optimize \( R \) to find a good pointwise estimate for \( I(f)(x) \).

Problem 3. Let \( X \) be a complex Hilbert space.

(a) Show that every orthonormal set \( A \subset X \) is contained in a maximal orthonormal set \( B \subset X \).

(b) Show that if \( (x_n) \) is an orthonormal set in \( X \), then \( x_n \to 0 \) in the weak topology on \( X \).

Hint: This follows from the more general condition in Problem 4, but derive this from Bessel’s inequality.

(c) Show that if \( X \) is separable and infinite-dimensional, then each maximal orthonormal set \( A \subset X \) is countably infinite.

Hint: Use the existence of a countably infinite maximal orthonormal set (Hilbert space basis) as discussed in class.
(d) Show that if $X$ is separable and infinite-dimensional, then there exists a linear isomorphism $T : X \rightarrow l^2$ that preserves the inner product.

**Problem 4.** Let $X$ be a complex separable Hilbert space with a Hilbert space basis $(x_n)$.

(a) Show that the infinite series $\sum_{n} \alpha_n x_n$ with coefficients $\alpha_n \in \mathbb{C}$ converges in $X$ if and only if $\sum_n |\alpha_n|^2 < \infty$.

(b) Show that for a sequence $(y_n)$ in $X$, we have $y_n \to y \in X$ in the weak topology on $X$ if and only if
   
   (i) there exists a constant $C$ such that $\|y_n\| \leq C$ for all $n$;
   
   (ii) $\langle y_n, x_k \rangle \to \langle y, x_k \rangle$ as $n \to \infty$ for each $k$.

(c) Let $(y_n)$ be a sequence in $X$. Show that then there exists $y \in X$ such that $y_n \overset{w}{\to} y$ if and only if
   
   (i) there exists a constant $C$ such that $\|y_n\| \leq C$ for all $n$;
   
   (ii) the sequence $(\langle y_n, x \rangle)$ converges for each $x \in X$. 


HOMEWORK C2

Problem 1. Let $X$ be a (complex) Hilbert space.

(a) Show that if $Y$ is another Hilbert space and $T : X \to Y$ is a bounded linear operator, then there exists a bounded linear operator $T^* : Y \to X$ (the adjoint of $T$) such that

$$\langle T(x), y \rangle = \langle x, T^* y \rangle$$

for all $x \in X$ and $y \in Y$.

*Hint: For fixed $y \in Y$, consider the map $x \mapsto \langle T(x), y \rangle$."

(b) Let $M$ be a closed subspace of $X$. Each $x \in X$ can be uniquely represented in the form $x = y + z$ with $y \in M$ and $z \in M^\perp$. Show that $x \mapsto y$ defines a bounded linear operator $P : X \to X$ (the orthogonal projection of $X$ to $M$) satisfying $P^2 = P$ and $P^* = P$. What is the operator norm $\|P\|$?

(c) Let $P : X \to X$ be a bounded linear operator with $P^2 = P$ and $P^* = P$. Show that then there exists a closed linear subspace $M$ of $X$ such that $P$ is the orthogonal projection of $X$ to $M$.

(d) Let $P_N$ denote the space of trigonometric polynomials of degree at most $N$, i.e. the set of all functions of the form

$$f = \sum_{n=-N}^{N} c_n u_n,$$

where $u_n(t) = e^{int}$ and $c_n \in \mathbb{C}$.

Show that $P_N$ is a closed linear subspace of $L^2(\mathbb{T})$ and that $s_N : L^2(\mathbb{T}) \to L^2(\mathbb{T})$ defined as

$$s_N(f) = \sum_{n=-N}^{N} \hat{f}(n) u_n$$

for $f \in L^2(\mathbb{T})$ is the orthogonal projection of $L^2(\mathbb{T})$ to $P_N$.

Problem 2. (a) Let $(c_n)_{n \in \mathbb{Z}}$ be a sequence of complex numbers with $\sum_n |c_n| < \infty$. Show that then $f(t) = \sum_n c_n e^{int}$ converges uniformly for $t \in \mathbb{R}$ and represents a $2\pi$-periodic continuous function $f$ on $\mathbb{R}$ with $\hat{f}(n) = c_n$.

(b) Suppose $f : \mathbb{R} \to \mathbb{C}$ is a $2\pi$-periodic function in $C^1(\mathbb{R})$. Show that then $\hat{f}(n) = O(1/n)$ as $|n| \to \infty$.

(c) Show that a $2\pi$-periodic function $f : \mathbb{R} \to \mathbb{C}$ is $C^\infty$-smooth if and only if for all $k$, we have $f^{(k)}(n) = O(1/|n|^k)$.

*Hint: For one of the directions, show that under suitable assumptions, the series in part (a) can be differentiated term-by-term."
**Problem 3.** Show that there are functions \( f \in C(\mathbb{T}) \) such that for the \( N \)-th partial sum \( s_N(f) \) of its Fourier series, we have \( s_N(f)(0) \not\to f(0) \) as \( N \to \infty \).

*Hint: Argue by contradiction. Consider the operators \( \Lambda_N : C(\mathbb{T}) \to \mathbb{C} \) given by \( \Lambda_N(f) = s_N(f)(0) \). Show that for their operator norms, we have \( \|\Lambda_N\| \geq \|D_N\|_1 \), where \( D_N \) is the Dirichlet kernel.*

**Problem 4.** Let \( \alpha \in \mathbb{R} \) be an irrational number.

(a) Suppose \( f : \mathbb{R} \to \mathbb{C} \) is a continuous function satisfying \( f(t + 1) = f(t) \) for \( t \in \mathbb{R} \). Show that then

\[
\frac{1}{N} \sum_{n=1}^{N} f(n\alpha) \xrightarrow{N \to \infty} \int_{0}^{1} f(t) \, dt.
\]

*Hint: First show this for certain types of functions \( f \) for which the sum on the left hand side can be computed explicitly.*

(b) For \( x \in \mathbb{R} \), let \( \text{frac}(x) \) denote the fractional part of \( x \). Prove Weyl’s equidistribution theorem: For each interval \([a, b] \subset [0, 1] \),

\[
\lim_{N \to \infty} \frac{\# \{ n \in \{1, \ldots, N\} \mid \text{frac}(n\alpha) \in [a, b] \} }{N} = b - a.
\]

(c) Show that \( \{ \text{frac}(n\alpha) \mid n \in \mathbb{N} \} \) is dense in \([0, 1]\).