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Problem 1. For $n \in \mathbb{N}$, we define the kernel $K_n : \mathbb{R} \to [0, \infty)$ by setting $K_n(x) = c_n(1 - x^2)^n$ for $|x| \leq 1$ and $K_n(x) = 0$ for $|x| > 1$. Here $c_n > 0$ is chosen so that $\int K_n(x) \, dx = 1$.

(a) Show that for each $\delta > 0$, we have $K_n \to 0$ as $n \to \infty$ uniformly on $\mathbb{R} \setminus (-\delta, \delta)$.

(b) Suppose that $f \in C_c(\mathbb{R})$ and $\text{supp } f \subset [0, 1]$. Define

$$P_n(x) = (K_n * f)(x) = \int K_n(x - u) f(u) \, du$$

for $x \in \mathbb{R}$. Show that for $x \in [0, 1]$, the expression $P_n(x)$ is equal to a polynomial in $x$.

(c) Show that we have uniform convergence $P_n \to f$ as $n \to \infty$ on each compact set $M \subset (0, 1)$.

(d) Use the previous considerations to prove Weierstrass’s approximation theorem: If $[a, b] \subset \mathbb{R}$ is a compact interval, then the set of polynomials is dense in $C([a, b])$.

Hint: First prove this for $[a, b] \subset (0, 1)$.

Problem 2. Let $n \geq 2$ and $p \in (1, n)$. For a function $f \in L^p(\mathbb{R}^n)$, we consider the Riesz potential

$$I(f)(x) = \int_{\mathbb{R}^n} \frac{|f(y)|}{|x - y|^{n-1}} \, dy, \quad x \in \mathbb{R}^n,$$

where $dy$ indicates integration with respect to Lebesgue measure.

(a) Fix $x \in \mathbb{R}^n$ and suppose that for some $R > 0$, we have $f = 0$ on $\mathbb{R}^n \setminus B(x, R)$. Show that then $I(f)(x) \leq C_1 R \cdot Mf(x)$, where $C_1 = C_1(n) > 0$ and $Mf$ denotes the (uncentered) Hardy-Littlewood maximal function of $f$.

Hint: Decompose $B(x, R)$ into dyadic annuli.

(b) Fix $x \in \mathbb{R}^n$. Suppose that for some $R > 0$, we have $f = 0$ on $B(x, R)$. Show that then $I(f)(x) \leq C_2 R^{1-n/p} \|f\|_p$, where $C_2 = C_2(p, n) > 0$.

(c) Show that $\|I(f)\|_{p^*} \leq C_3 \|f\|_p$, where $p^* = np/(n - p)$ and $C_3 = C_3(p, n) > 0$.

Hint: Split the given function $f$ into two functions as suggested by (a) and (b). Optimize $R$ to find a good pointwise estimate for $I(f)(x)$.

Problem 3. Let $X$ be a complex Hilbert space.

(a) Show that every orthonormal set $A \subset X$ is contained in a maximal orthonormal set $B \subset X$.

(b) Show that if $(x_n)$ is an orthonormal set in $X$, then $x_n \to 0$ in the weak topology on $X$.

Hint: This follows from the more general condition in Problem 4, but derive this from Bessel’s inequality.

(c) Show that if $X$ is separable and infinite-dimensional, then each maximal orthonormal set $A \subset X$ is countably infinite.

Hint: Use the existence of a countably infinite maximal orthonormal set (Hilbert space basis) as discussed in class.
(d) Show that if $X$ is separable and infinite-dimensional, then there exists a linear isomorphism $T : X \rightarrow l^2$ that preserves the inner product.

**Problem 4.** Let $X$ be a complex separable Hilbert space with a Hilbert space basis $(x_n)$.

(a) Show that the infinite series $\sum_n \alpha_n x_n$ with coefficients $\alpha_n \in \mathbb{C}$ converges in $X$ if and only if $\sum_n |\alpha_n|^2 < \infty$.

(b) Show that for a sequence $(y_n)$ in $X$, we have $y_n \rightarrow y \in X$ in the weak topology on $X$ if and only if

(i) there exists a constant $C$ such that $\|y_n\| \leq C$ for all $n$;

(ii) $\langle y_n, x_k \rangle \rightarrow \langle y, x_k \rangle$ as $n \rightarrow \infty$ for each $k$.

(c) Let $(y_n)$ be a sequence in $X$. Show that then there exists $y \in X$ such that $y_n \overset{w}{\rightarrow} y$ if and only if

(i) there exists a constant $C$ such that $\|y_n\| \leq C$ for all $n$;

(ii) the sequence $(\langle y_n, x \rangle)$ converges for each $x \in X$. 