Contents

1 Homework 1 3
2 Homework 2 5
3 Homework 3 7
 HOMEWORK 1

Problem 1. Let $A$ be a positive definite $n \times n$ matrix. Show that

$$\int_{\mathbb{R}^n} \exp(-x^tAx) \, d\lambda_n(x) = \frac{\pi^{n/2}}{(\det A)^{1/2}}.$$  

Here $\lambda_n$ denotes Lebesgue measure on $\mathbb{R}^n$ and $x^t$ is the transpose of the column vector $x \in \mathbb{R}^n$.

*Hint: Use the transformation formula and Fubini-Tonelli.*

Problem 2. The purpose of this problem is to give a simple application of the Whitney cube decomposition of open sets. Stronger results than in this problem can be proved by other methods that we will discuss later.

Let $K \subset \mathbb{R}^n$ be a compact set which is porous. This means that there exists a constant $c \in (0, 1)$ with the following property: for each $x \in K$ and each $r > 0$, there exists $y \in B(x, r)$ such that $B(y, cr) \subset \mathbb{R}^n \setminus K$. In other words, if we consider a ball $B$ centered at a point in $K$, then $K$ has a “hole” in $B$ of size comparable to $B$.

(a) Show that $\lambda_n(K) = 0$.

*Hint: Fix a large $R > 0$ such that $K \subset B(0, R)$. Consider a Whitney cube decomposition of $U = B(0, R) \setminus K$. Show that there is a factor $\lambda = \lambda(c, n) > 0$ such that if we enlarge each Whitney cube by this factor $\lambda$, then every point in $K$ is contained in infinitely many enlarged Whitney cubes. Show that, on the other hand, the sum of the characteristic functions of the enlarged cubes is in $L^1(\mathbb{R}^n)$.*

(b) Formulate a condition for the compact set $K$ that is weaker than porosity, but similar in spirit and still ensures the conclusion $\lambda_n(K) = 0$.

Problem 3. Let $(X, \mathcal{A})$ be a measurable space.

(a) Let $\mu$ be a signed measure on $(X, \mathcal{A})$ and $\mu = \mu^+ - \mu^-$ be its Jordan decomposition. Show that if $\lambda$ and $\nu$ are positive measures on $(X, \mathcal{A})$ with $\mu = \lambda - \nu$, then $\lambda \geq \mu^+$ and $\nu \geq \mu^-$, i.e. $\lambda(A) \geq \mu^+(A)$ and $\nu(A) \geq \mu^-(A)$ for all $A \in \mathcal{A}$.

(b) Let $\mu$ be a signed measure on $(X, \mathcal{A})$ and $|\mu|$ be its total variation. Show that

$$|\mu|(A) = \sup \left\{ \sum_{n=1}^{\infty} |\mu(B_n)| \mid B_n \in \mathcal{A} \text{ pairwise disjoint and } \bigcup_{n=1}^{\infty} B_n = A \right\}$$

for each $A \in \mathcal{A}$.

(c) Let $\mu$ and $\nu$ be signed measures on $(X, \mathcal{A})$ that both omit $+\infty$ or both omit $-\infty$. Show that then $\mu + \nu$ is a signed measure on $(X, \mathcal{A})$ with $|\mu + \nu| \leq |\mu| + |\nu|$.

Problem 4 (Analysis Spring 2012). (a) Suppose $f : [0, 1) \to \mathbb{C}$ is integrable with respect to Lebesgue measure on $[0, 1)$. For $n \in \mathbb{N}$, define

$$f_n(x) = n \int_{(k-1)/n}^{k/n} f(t) \, dt \quad \text{if } x \in \left[ \frac{k-1}{n}, \frac{k}{n} \right),$$

where $k \in \{1, \ldots, n\}$. Show that $f_n \to f$ in $L^1([0, 1))$ as $n \to \infty$. 


(b) Let $S$ be the set of all complex-valued functions $f$ on $\mathbb{R}^3$ with

$$f \in L^1(\mathbb{R}^3) \cap L^2(\mathbb{R}^3) \quad \text{and} \quad \int f \, d\lambda_3 = 0.$$ 

Show that $S$ is dense in $L^2(\mathbb{R}^3)$. 
HOMEWORK 2

Problem 1. Let \((X, \mathcal{A})\) be a measurable space. We denote by \(\mathcal{M}\) the set of all finite signed measures on \((X, \mathcal{A})\).

(a) If \(a, b \in \mathbb{R}\) and \(\mu, \nu \in \mathcal{M}\), we define
\[(a\mu + b\nu)(A) = a\mu(A) + b\nu(A)\]
for \(A \in \mathcal{A}\). Show that \(a\mu + b\nu \in \mathcal{M}\) and that \(\mathcal{M}\) is a vector space over \(\mathbb{R}\) with this linear structure.

(b) For \(\mu \in \mathcal{M}\), define
\[\|\mu\| = |\mu|(X)\]
Show that \(\mu \mapsto \|\mu\|\) defines a norm on \(\mathcal{M}\).

(c) Show that the vector space \(\mathcal{M}\) equipped with the norm defined in (b) is a Banach space.

Problem 2. Let \(\nu\) be a complex measure on a measurable space \((X, \mathcal{A})\).

(a) Let \(\nu^+_r = \nu^+_r - \nu^-_r\) and \(\nu^+_i = \nu^+_i - \nu^-_i\) be the Jordan decompositions of the real part \(\nu^+_r\) and the imaginary part \(\nu^+_i\) of \(\nu\). Show that if \(|\nu|\) denotes the total variation of \(\nu\), then
\[\nu^+_r, \nu^-_r, \nu^+_i, \nu^-_i \leq |\nu|\]
(b) We say that a measurable function \(f\) on \((X, \mathcal{A})\) is \(\nu\)-integrable if it is integrable with respect to \(|\nu|\), so if we denote the space of these functions \(f\) by \(L^1(\nu)\), then \(L^1(\nu) = L^1(|\nu|)\). Show that if \(f \in L^1(\nu)\), then \(f\) is integrable with respect to each of the measures \(\nu^+_r, \nu^-_r, \nu^+_i, \nu^-_i\), and so
\[
\int f \, d\nu = \int f \, d\nu^+_r - \int f \, d\nu^-_r + i \int f \, d\nu^+_i - i \int f \, d\nu^-_i
\]
is well-defined.

(c) Suppose \(\mu\) is a \(\sigma\)-finite positive measure on \((X, \mathcal{A})\) such that \(\nu \ll \mu\) and let \(g = d\nu/d\mu\) be the Radon-Nikodym derivative of \(\nu\) with respect to \(\mu\). Show that if \(f \in L^1(\nu)\), then \(fg \in L^1(\mu)\) and
\[
\int f \, d\nu = \int fg \, d\mu.
\]

Problem 3. (a) Let \(f : \mathbb{R}^n \to \mathbb{C}\) and \(g : \mathbb{R}^n \to \mathbb{C}\) be Borel measurable functions on \(\mathbb{R}^n\). Show that the function \(F : \mathbb{R}^{2n} = \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}\) defined as
\[F(x, y) = f(x - y)g(y), \quad x, y \in \mathbb{R}^n\]
is also Borel measurable.

(b) Let \(f : \mathbb{R}^n \to \mathbb{C}\) and \(g : \mathbb{R}^n \to \mathbb{C}\) be (Lebesgue) integrable functions. Show that then the convolution of \(f\) and \(g\), given by
\[(f * g)(x) = \int_{\mathbb{R}^n} f(x - y)g(y) \, d\lambda_n(y),\]
is well-defined for almost every \(y \in \mathbb{R}^n\) and
\[\|f * g\|_1 \leq \|f\|_1 \cdot \|g\|_1.\]
(c) Show that if $f : \mathbb{R}^n \to \mathbb{C}$ and $g : \mathbb{R}^n \to \mathbb{C}$ are integrable functions, then $(f * g)(x) = (g * f)(x)$ for almost every $x \in \mathbb{R}^n$.

**Problem 4.** Let $X$ be a topological space and $f : X \to X$ be a continuous map. A Borel measure $\mu$ on $X$ is called $f$-invariant if $f_* \mu = \mu$, or equivalently, if $\mu(f^{-1}(B)) = \mu(B)$ for all Borel sets $B \subset X$.

Consider two $f$-invariant Borel probability measures $\nu$ and $\mu$ on $X$ and let $\nu = \nu_s + \nu_a$ be the Lebesgue decomposition of $\nu$ with respect to $\mu$, where $\nu_s \perp \mu$ and $\nu_a \ll \mu$. Show that then $\nu_s$ and $\nu_a$ are also $f$-invariant.
HOMEWORK 3

Problem 1. (a) Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be two norms on $\mathbb{R}^n$. Show that these norms are equivalent, i.e. there exists a constant $C \geq 1$ such that $C^{-1}\|x\|_1 \leq \|x\|_2 \leq C\|x\|_1$ for all $x \in \mathbb{R}^n$.

(b) Let $X$ be a real normed vector space and $T : \mathbb{R}^n \to X$ be linear. Show that $T$ is bounded.

(c) Let $X$ be a real normed vector space and $U \subset X$ be a finite-dimensional subspace of $X$. Show that $U$ is closed in $X$.

Problem 2. A bump function on $\mathbb{R}^n$ is a $C^\infty$-smooth $\varphi : \mathbb{R}^n \to \mathbb{R}$ with compact support such that $\varphi \geq 0$ and $\varphi \neq 0$. The purpose of this problem is to show the existence of certain bump functions.

(a) Let $f(x) = \exp(-1/x^2)$ for $x > 0$ and $f(x) = 0$ for $x \leq 0$. Show that $f$ is $C^\infty$-smooth on $\mathbb{R}$.

   Hint: Show that $f^{(n)}$ exists by induction on $n$. For this, it helps to find a general type of expression that represents $f^{(n)}$.

(b) Show that if $f$ is as in (a) and $a, b \in \mathbb{R}$ with $a < b$, then $g(x) = f(x - a)f(b - x)$ defines a bump function on $\mathbb{R}$ with support in the interval $[a, b]$.

(c) Show that if $a, b \in \mathbb{R}$ with $a < b$, then there exists a $C^\infty$-smooth function $h$ on $\mathbb{R}$ with $0 \leq h \leq 1$ such that $h(x) = 1$ for $x \leq 1$ and $h(x) = 0$ for $x \geq b$.

(d) Let $0 < r < R$. Show that there exists a $C^\infty$-smooth function $\varphi$ on $\mathbb{R}^n$ with $0 \leq \varphi \leq 1$ such that $\varphi(x) = 1$ for $x \in \mathbb{R}^n$ with $\|x\| \leq r$ and $\varphi(x) = 0$ for $x \in \mathbb{R}^n$ with $\|x\| \geq R$.

Problem 3. Let $V$ be a (possibly infinite-dimensional) vector space over a field $F$.

(a) Use Zorn’s lemma to show that $V$ contains a maximal linearly independent subset $I$.

(b) Show that a set $I$ as in (a) is a basis of $V$.

Problem 4. (a) Let $X$ be a normed vector space and $U \neq X$ be a closed subspace. Show that then there exists a non-zero functional $f \in X^*$ such that $f|_U = 0$.

   Hint: Pick $x \in X \setminus U$, consider the span of $U$ and $x$, and apply the Hahn-Banach theorem.

(b) Let $X$ be a normed vector space. Suppose there exist vectors $x_n$ for $n \in \mathbb{N}$ whose span is dense in $X$. Show that then $X$ is separable.

(c) Let $X$ be a normed vector space and $X^*$ be its dual space. Show that if $X^*$ is separable, then $X$ is also separable.

   Hint: Let $\{f_n \mid n \in \mathbb{N}\}$ be a countable dense subset of $X^*$. Then we can find $x_n \in X$ with $\|x_n\| \leq 1$ such that $|f_n(x_n)| \geq \|f_n\|/2$.

   Remark: Part (c) is Problem 6 from the Fall 2014 Analysis Qual (the hint was not given).