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HOMEWORK 1

Problem 1. Construct a smooth manifold structure on the Grassmannian $Gr_n(\mathbb{R}^k)$, the set of all $n$-dimensional subspaces of $\mathbb{R}^k$. Construct the canonical $n$-dimensional vector bundle over it (as a smooth vector bundle, not just the fiber over each point).

Problem 2. Let $M$ be a compact $n$-dimensional manifold and $f : M \to \mathbb{R}^n$ be a smooth map. Prove that $f$ is singular (that is, $df$ has rank less than $n$) somewhere.

Problem 3. A Riemannian structure on a smooth manifold is a choice of a positive definite inner product $\langle \cdot, \cdot \rangle_p$ on each tangent space $T_pM$, which is smooth in the sense that whenever $X$ and $Y$ are two smooth vector bundles, $\langle X, Y \rangle$ is smooth. Prove that there is a Riemannian structure on every smooth manifold.

Hint: Use partitions of unity.

Problem 4. Prove that there are exactly two isomorphism classes of line bundles (one-dimensional vector bundles) over $S^1$. Which of them is the tangent bundle $T^*S^1$?

Problem 5. If $A \subset M$ is a closed submanifold and $U \supset A$ is any open neighborhood, and $f : A \to \mathbb{R}$ is a smooth function, prove that there is a smooth function $g : M \to \mathbb{R}$ with $g|_A = f$ and $g = 0$ outside $U$.

Problem 6. Consider the maximal atlas on $\mathbb{R}$ containing the function $t \mapsto t^3$. Prove that it produces a smooth structure on $\mathbb{R}$ distinct but diffeomorphic to the standard smooth structure on $\mathbb{R}$. (The standard smooth structure comes from the maximal atlas containing the function $t \mapsto t$.)
HOMEWORK 2

Problem 1. View $S^2$ as the unit sphere in $\mathbb{R}^3$.

(a) Let $p_0 \in S^2$ be the point $(0, 0, 1)$. If $SO(3)$ is the special orthogonal group, define $f : SO(3) \to S^2$ by $f(A) = A(p_0)$. Prove that this is a smooth fiber bundle with fiber $SO(2) \cong S^1$.

(b) The tangent bundle $T^*S^2$ carries a Riemannian metric coming from the dot product in $\mathbb{R}^3$. Let $ST^*S^2$ be its unit sphere bundle, which is a fiber bundle over $S^2$ whose fibers over each point are the length-one tangent vectors over that point. Construct a diffeomorphism between $ST^*S^2$ and $SO(3)$ as fiber bundles over $S^2$.

(c) Consider the Gauss map $S^2 \to Gr_2(\mathbb{R}^3) \cong \mathbb{RP}^2$. Prove that this is the orientation double cover.

Problem 2. Let $f : M \to N$, and suppose $(x, U)$ and $(y, V)$ are coordinate systems around $p$ and $f(p)$, respectively.

(a) If $g : N \to \mathbb{R}$, then prove that

$$\frac{\partial (g \circ f)}{\partial x^i}(p) = \sum_j \frac{\partial g}{\partial y^j}(f(p)) \frac{\partial (y^j \circ f)}{\partial x^i}(p)$$

(b) Prove that

$$f_* \left( \frac{\partial}{\partial x^i} \bigg|_p \right) = \sum_j \frac{\partial (y^j \circ f)}{\partial x^i}(p) \frac{\partial}{\partial y^j} \bigg|_{f(p)},$$

and more generally, express $f_* \left( \sum_i a^i \frac{\partial}{\partial x^i} \bigg|_p \right)$ in terms of $\frac{\partial}{\partial y^j} \big|_{f(p)}$.

(c) Show that

$$(f^* dy^j)(p) = \sum_i \frac{\partial (y^j \circ f)}{\partial x^i}(p) dx^i(p).$$

(d) More generally, express

$$f^* \left( \sum_{j_1, \ldots, j_k} a_{j_1, \ldots, j_k} dy^{j_1} \otimes \cdots \otimes dy^{j_k} \right)$$

in terms of the $dx^i$.

Problem 3. (a) Construct isomorphisms $\phi_V : V \to V^{**}$ for all finite-dimensional vector spaces $V$ so that for any linear map $f : V \to W$, the following commutes.

$$\begin{array}{ccc}
V & \xrightarrow{f} & W \\
\phi_V & \downarrow & \downarrow \phi_W \\
V^{**} & \xrightarrow{f^{**}} & W^{**}
\end{array}$$
(b) Prove that there do not exist isomorphisms $\phi_V : V \to V^*$ for all finite-dimensional vector spaces $V$ so that for any linear map $f : V \to W$, the following commutes.

\[
\begin{array}{ccc}
V & \xrightarrow{f} & W \\
\downarrow{\phi_V} & & \downarrow{\phi_W} \\
V^* & \xleftarrow{f^*} & W^*
\end{array}
\]

(c) Construct isomorphisms $\phi_V : V \to V^*$ for all finite-dimensional inner product spaces $V$ so that for any linear map $f : V \to W$ between inner product spaces that preserves inner products, the following commutes.

\[
\begin{array}{ccc}
V & \xrightarrow{f} & W \\
\downarrow{\phi_V} & & \downarrow{\phi_W} \\
V^* & \xleftarrow{f^*} & W^*
\end{array}
\]

**Problem 4.** Prove that the tangent bundle and the cotangent bundle of a manifold are always isomorphic (although not canonically).
HOMEWORK 3

Problem 1. Let $M_n$ be the space of $n \times n$ real matrices and let $M_n^k$ be the subspace of all matrices of rank $k$. Prove that $M_n^k$ is a submanifold of $M_n$.

*Hint:* Fix some $k \times k$ minor and consider the subspace of $M_n$ where this minor has non-zero determinant.

Problem 2. The $n$-dimensional torus $T^n$ is defined to be $\mathbb{R}^n/\mathbb{Z}^n$, i.e. for any $x, y \in \mathbb{R}^n$, we say that $x \sim y$ if and only if $x - y \in \mathbb{Z}^n$. Let $\alpha, \beta : \mathbb{R}^n \to \mathbb{R}$ be two nowhere zero functions such that

(i) $\alpha(x) = \alpha(y)$ and $\beta(x) = \beta(y)$ if $x - y \in \mathbb{Z}^n$, and

(ii) $\alpha/\beta$ is an irrational constant.

Then the vector field $\alpha(x) \frac{\partial}{\partial x^1} + \beta(x) \frac{\partial}{\partial x^2}$ on $\mathbb{R}^n$ descends to a vector field $X$ on $T^n$. Find all functions $f : T^n \to \mathbb{R}$ such that $Xf = 0$.

Problem 3. An $n$-manifold is called parallelizable if one can find $n$ vector fields which are linearly independent at each point.

(a) Prove that $S^3$ is parallelizable.

(b) Prove that $S^1 \times S^2$ is parallelizable.

(c) Prove that $S^1 \times S^n$ is parallelizable.

*Hint:* $S^n$ is the unit sphere in $\mathbb{R}^{n+1}$, and $\mathbb{R}^{n+1}$ is parallelizable.

Problem 4. Let $M$ be a connected smooth manifold. Show that for any points $x, y \in M$, there is a diffeomorphism $f : M \to M$ such that $f(x) = y$.

Problem 5. View $S^n$ as the unit sphere in $\mathbb{R}^{n+1}$; the restriction of the standard metric on $\mathbb{R}^{n+1}$ makes $S^n$ a Riemannian manifold. Consider the stereographic projection

$$x : U = S^n \setminus \{(0, \ldots, 0, 1)\} \to \mathbb{R}^n, \quad (p_1, \ldots, p_n, p_{n+1}) \mapsto \left(\frac{p_1}{1 - p_{n+1}}, \ldots, \frac{p_n}{1 - p_{n+1}}\right).$$

Write down the metric on $U$ explicitly as $\sum_{i,j} g_{ij} dx^i \otimes dx^j$ in terms of these local coordinates.
HOMEWORK 4

Problem 1. Construct a vector field on $\mathbb{R}$ whose integral curves through any point are only defined for finite time.

Problem 2. For any (smooth) vector field $X$ and any (smooth) function $f$ on a manifold $M$, prove

$$L_X(df) = d(L_X f).$$

Problem 3. Consider the vector fields on $\mathbb{R}^3$

$$X = z \frac{\partial}{\partial y} - y \frac{\partial}{\partial z},$$
$$Y = x \frac{\partial}{\partial z} - z \frac{\partial}{\partial x},$$
$$Z = y \frac{\partial}{\partial x} - x \frac{\partial}{\partial y}.$$

For any vector $v = (a, b, c) \in \mathbb{R}^3$, consider the vector field

$$W(v) = aX + bY + cZ.$$

(a) Prove that cross product corresponds to the Lie bracket,

$$W(u \times v) = [W(u), W(v)].$$

(b) Geometrically describe the flow of $W(v)$ in terms of $v$.

Problem 4. For a vector field $X$ and an $(l, k)$-tensor field $A$, define the Lie derivative $L_X A$ as another $(l, k)$-tensor field so that

(i) for any two $(l, k)$-tensor fields $A$ and $B$, we have $L_X (A + B) = L_X A + L_X B$;

(ii) for any two vector fields $X$ and $Y$, we have $L_{X+Y} A = L_X A + L_Y A$;

(iii) for any $(l, k)$-tensor field $A$ and $(l', k')$-tensor field $B$, we have $L_X (A \otimes A') = (L_X A) \otimes A' + A \otimes (L_X A')$;

(iv) if $C : T^k_l \to T^{l'-1}_{l-1}$ denotes any fixed contraction, we have $L_X (CA) = C(L_X A)$;

(v) for any vector fields $X_1, \ldots, X_k$ and 1-forms $\omega_1, \ldots, \omega_l$, we have

$$L_X (A(X_1, \ldots, X_k, \omega_1, \ldots, \omega_l)) = (L_X A)(X_1, \ldots, X_k, \omega_1, \ldots, \omega_l)$$
$$+ \sum_{i=1}^{k} A(X_1, \ldots, L_X X_i, \ldots, X_k, \omega_1, \ldots, \omega_l)$$
$$+ \sum_{j=1}^{l} A(X_1, \ldots, X_k, \omega_1, \ldots, L_X \omega_j, \ldots, \omega_l).$$
If
\[ A = \sum A_{i_1 \ldots i_k} \frac{\partial}{\partial x^{i_1}} \ldots \frac{\partial}{\partial x^{i_k}} \]

in local coordinates, then write down \( L_X A \) in local coordinates.

**Problem 5.** If \( f : M \to N \) is a surjective map that is regular everywhere and \( M \) is compact, then prove that \( f \) is a smooth fiber bundle.

*Hint: It is enough to consider \( N = \mathbb{R}^n \). For the case \( N = \mathbb{R} \), construct the diffeomorphism \( f^{-1}(U) \cong f^{-1}(q) \times U \) by flowing along some vector field.*