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HOMEWORK 1

Problem 1. Determine all subrings of $\mathbb{Z}$.

Problem 2. Determine all subrings of $\mathbb{Z} \times \mathbb{Z}$.

Problem 3. Determine all ideals of $\mathbb{Z} \times \cdots \times \mathbb{Z}$ ($n$ times).

Problem 4. Give an example of a commutative ring $R$ and two distinct ideals $I$ and $J$ of $R$ such that $I \cap J \neq IJ$.

Problem 5. Determine all finite rings of 2 and 3 elements.

Problem 6. Let $R$ be a commutative ring and $r \in R$. Prove there is a unique ring homomorphism $f : \mathbb{Z}[x] \to R$ such that $f(x) = r$. Show that the image of $f$ is the smallest subring of $R$ that contains $r$.

Problem 7. Let $R$ be an integral domain such that $R[x]$ is a principal ideal domain. Prove that $R$ is a field.

Problem 8. Prove that for every non-zero commutative ring $R$, the ring $R[x]$ has infinitely many prime ideals.

Problem 9. Let $B \subset A$ be a subgroup of an abelian group $A$. Prove that the set

$$I = \{ f \in \text{End } A \mid f(A) \subset B \}$$

is a right ideal in the ring $\text{End } A$.

Problem 10. The Jacobson radical $\text{rad } R$ (or $J(R)$) of a commutative ring $R$ is the intersection of all maximal ideals of $R$. Show that $x \in \text{rad } R$ if and only if $1 - xy \in R^\times$ for all $y \in R$. 
HOMEWORK 2

Problem 1. Let \( f \) be a polynomial over a commutative ring \( R \). Prove that if \( f \in \text{rad}(R[x]) \), then \( f \) is a nilpotent polynomial.

Problem 2. (a) Let \( R = R_1 \times \cdots \times R_n \) be the product of rings. For every \( i = 1, \ldots, n \), let \( e_i = (0, \ldots, 0, 1, 0, \ldots, 0) \), with 1 in the \( i \)-th position. Prove the following:
   (i) \( e_i^2 = e_i \) for all \( i \);
   (ii) \( e_i e_j = 0 \) for all \( i \neq j \);
   (iii) \( 1 = e_1 + \cdots + e_n \);
   (iv) \( e_i a = ae_i \) for all \( i \) and \( a \in R \).

(b) Let \( R \) be a ring and \( e_1, \ldots, e_n \in R \) satisfy (i)-(iv) as above. Prove that \( R \cong R_1 \times \cdots \times R_n \) for some rings \( R_i \) so that \( e_i \) correspond to \( (0, \ldots, 0, 1, 0, \ldots, 0) \) under the isomorphism.

Problem 3. Let \( e \) be an idempotent in a commutative ring \( R \), i.e. \( e^2 = e \). Prove that if \( R \) is a local ring, i.e. \( R \) has exactly one maximal ideal, then \( e \in \{0, 1\} \).

Problem 4. Let \( R \) be a commutative ring such that every \( a \in R \) is idempotent. Prove that every prime ideal in \( R \) is maximal.

Problem 5. Prove that every commutative non-zero ring has a minimal prime ideal.

Problem 6. Determine the spectrum of the subring \( R \subset \mathbb{Q} \) of all fractions \( a/b \) with \( b \) odd.

Problem 7. Prove that for every two commutative rings \( R \) and \( S \), the spectrum of the product \( R \times S \) is bijective to the disjoint union of \( \text{Spec} R \) and \( \text{Spec} S \).

Problem 8. Let \( R \) be a commutative ring and \( I \subset R \) an ideal. Construct a bijection between \( \text{Spec}(R/I) \) and the subset of all prime ideals \( P \in \text{Spec} R \) containing \( I \).

Problem 9. Let \( p \) be a divisor of \( 4a^2 + 1 \) for some integer \( a \). Prove that \( p \equiv 1 \pmod{4} \).

Problem 10. Let \( p \) be a prime integer with \( p \equiv 1 \pmod{4} \). Prove that \( p = a^2 + b^2 \) for a unique (non-ordered) pair of positive integers \( (a, b) \).
HOMEWORK 3

Problem 1. Show that for every field $F$, the set Spec $F[x]$ is infinite.

Problem 2. Show that $\mathbb{Z}[\sqrt{2}]$ is a UFD.

Problem 3. Find an ideal of $\mathbb{Z}[\sqrt{-5}]$ that is not principal.

Problem 4. Prove that every non-zero prime ideal of a PID is maximal.

Problem 5. Let $R$ be a commutative ring and $S \subseteq R$ be a multiplicative subset, meaning that $1 \in S$ and that $S$ is closed under multiplication. Define a relation $\sim$ on the set $R \times S$ by the condition that $(r_1, s_1) \sim (r_2, s_2)$ if and only if there is an element $s \in S$ such that $s(r_1s_2 - r_2s_1) = 0$.

(a) Prove that $\sim$ is an equivalence relation. Write $r/s$ for the equivalence class of $(r, s)$ and $S^{-1}R$ for the set of equivalence classes.

(b) Prove that the binary operations

$$\frac{r_1}{s_1} + \frac{r_2}{s_2} = \frac{r_1s_2 + r_2s_1}{s_1s_2} \quad \text{and} \quad \frac{r_1}{s_1} \cdot \frac{r_2}{s_2} = \frac{r_1r_2}{s_1s_2}$$

are well-defined and make $S^{-1}R$ a commutative ring.

(c) Prove that the map $f : R \to S^{-1}R$ taking $r$ to $r/1$ is a ring homomorphism.

Problem 6. Let $R$ be a commutative ring and $S \subseteq R$ be a multiplicative subset. Let $g : R \to T$ be a homomorphism of commutative rings such that $g(S) \subseteq T^\times$. Prove that there is a unique ring homomorphism $h : S^{-1}R \to T$ such that $g = h \circ f$, where $f : R \to S^{-1}R$ is the homomorphism from Problem 5(c).

Problem 7. Prove that if $R$ is a UFD and $0 \not\in S \subseteq R$ is a multiplicative subset, then $S^{-1}R$ is also a UFD.

Problem 8. Let $R$ be an integral domain with quotient field $F$. Show that if $T$ is an integral domain such that $R \subseteq T \subseteq F$, then the quotient field of $T$ is isomorphic to $F$.

Problem 9. Prove that the ring $R[x_1, x_2, \ldots]$ is not noetherian.

Problem 10. Show that the ring $R[x_1, x_2, \ldots]$ is a UFD.