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HOMEWORK 1

**Problem 1.** Determine all subrings of \( \mathbb{Z} \).

**Problem 2.** Determine all subrings of \( \mathbb{Z} \times \mathbb{Z} \).

**Problem 3.** Determine all ideals of \( \mathbb{Z} \times \cdots \times \mathbb{Z} \) (\( n \) times).

**Problem 4.** Give an example of a commutative ring \( R \) and two distinct ideals \( I \) and \( J \) of \( R \) such that \( I \cap J \neq IJ \).

**Problem 5.** Determine all finite rings of 2 and 3 elements.

**Problem 6.** Let \( R \) be a commutative ring and \( r \in R \). Prove there is a unique ring homomorphism \( f : \mathbb{Z}[x] \to R \) such that \( f(x) = r \). Show that the image of \( f \) is the smallest subring of \( R \) that contains \( r \).

**Problem 7.** Let \( R \) be an integral domain such that \( R[x] \) is a principal ideal domain. Prove that \( R \) is a field.

**Problem 8.** Prove that for every non-zero commutative ring \( R \), the ring \( R[x] \) has infinitely many prime ideals.

**Problem 9.** Let \( B \subset A \) be a subgroup of an abelian group \( A \). Prove that the set

\[
I = \{ f \in \text{End} A \mid f(A) \subset B \}
\]

is a right ideal in the ring \( \text{End} A \).

**Problem 10.** The Jacobson radical \( \text{rad} R \) (or \( J(R) \)) of a commutative ring \( R \) is the intersection of all maximal ideals of \( R \). Show that \( x \in \text{rad} R \) if and only if \( 1 - xy \in R^\times \) for all \( y \in R \).
HOMEWORK 2

Problem 1. Let $f$ be a polynomial over a commutative ring $R$. Prove that if $f \in \text{rad}(R[x])$, then $f$ is a nilpotent polynomial.

Problem 2. (a) Let $R = R_1 \times \cdots \times R_n$ be the product of rings. For every $i = 1, \ldots, n$, let $e_i = (0, \ldots, 0, 1, 0, \ldots, 0)$, with 1 in the $i$-th position. Prove the following:

(i) $e_i^2 = e_i$ for all $i$;
(ii) $e_ie_j = 0$ for all $i \neq j$;
(iii) $1 = e_1 + \cdots + e_n$;
(iv) $e_i a = ae_i$ for all $i$ and $a \in R$.

(b) Let $R$ be a ring and $e_1, \ldots, e_n \in R$ satisfy (i)-(iv) as above. Prove that $R \cong R_1 \times \cdots \times R_n$ for some rings $R_i$ so that $e_i$ correspond to $(0, \ldots, 0, 1, 0, \ldots, 0)$ under the isomorphism.

Problem 3. Let $e$ be an idempotent in a commutative ring $R$, i.e. $e^2 = e$. Prove that if $R$ is a local ring, i.e. $R$ has exactly one maximal ideal, then $e \in \{0, 1\}$.

Problem 4. Let $R$ be a commutative ring such that every $a \in R$ is idempotent. Prove that every prime ideal in $R$ is maximal.

Problem 5. Prove that every commutative non-zero ring has a minimal prime ideal.

Problem 6. Determine the spectrum of the subring $R \subset \mathbb{Q}$ of all fractions $a/b$ with $b$ odd.

Problem 7. Prove that for every two commutative rings $R$ and $S$, the spectrum of the product $R \times S$ is bijective to the disjoint union of Spec $R$ and Spec $S$.

Problem 8. Let $R$ be a commutative ring and $I \subset R$ an ideal. Construct a bijection between Spec$(R/I)$ and the subset of all prime ideals $P \in \text{Spec} R$ containing $I$.

Problem 9. Let $p$ be a divisor of $4a^2 + 1$ for some integer $a$. Prove that $p \equiv 1 \pmod{4}$.

Problem 10. Let $p$ be a prime integer with $p \equiv 1 \pmod{4}$. Prove that $p = a^2 + b^2$ for a unique (non-ordered) pair of positive integers $(a, b)$. 
HOMEWORK 3

Problem 1. Show that for every field $F$, the set $\text{Spec } F[x]$ is infinite.

Problem 2. Show that $\mathbb{Z}[\sqrt{2}]$ is a UFD.

Problem 3. Find an ideal of $\mathbb{Z}[\sqrt{-5}]$ that is not principal.

Problem 4. Prove that every non-zero prime ideal of a PID is maximal.

Problem 5. Let $R$ be a commutative ring and $S \subset R$ be a multiplicative subset, meaning that $1 \in S$ and that $S$ is closed under multiplication. Define a relation $\sim$ on the set $R \times S$ by the condition that $(r_1, s_1) \sim (r_2, s_2)$ if and only if there is an element $s \in S$ such that $s(r_1s_2 - r_2s_1) = 0$.

(a) Prove that $\sim$ is an equivalence relation. Write $r/s$ for the equivalence class of $(r, s)$ and $S^{-1}R$ for the set of equivalence classes.

(b) Prove that the binary operations

$$
\frac{r_1}{s_1} + \frac{r_2}{s_2} = \frac{r_1s_2 + r_2s_1}{s_1s_2} \quad \text{and} \quad \frac{r_1}{s_1} \cdot \frac{r_2}{s_2} = \frac{r_1r_2}{s_1s_2}
$$

are well-defined and make $S^{-1}R$ a commutative ring.

(c) Prove that the map $f : R \to S^{-1}R$ taking $r$ to $r/1$ is a ring homomorphism.

Problem 6. Let $R$ be a commutative ring and $S \subset R$ be a multiplicative subset. Let $g : R \to T$ be a homomorphism of commutative rings such that $g(S) \subset T^\times$. Prove that there is a unique ring homomorphism $h : S^{-1}R \to T$ such that $g = h \circ f$, where $f : R \to S^{-1}R$ is the homomorphism from Problem 5(c).

Problem 7. Prove that if $R$ is a UFD and $0 \not\in S \subset R$ is a multiplicative subset, then $S^{-1}R$ is also a UFD.

Problem 8. Let $R$ be an integral domain with quotient field $F$. Show that if $T$ is an integral domain such that $R \subset T \subset F$, then the quotient field of $T$ is isomorphic to $F$.

Problem 9. Prove that the ring $R = \mathbb{Z}[x_1, x_2, \ldots]$ is not noetherian.

Problem 10. Show that the ring $R = \mathbb{Z}[x_1, x_2, \ldots]$ is a UFD.
**HOMEWORK 4**

**Problem 1.** Give an example of a non-zero abelian group such that \( A \oplus A \cong A \).

**Problem 2.** Let \( A \) be a non-zero abelian group such that \( A \oplus A \cong A \) and let \( R = \text{End} A \). Prove that the free \( R \)-modules \( R^n \) and \( R^m \) are isomorphic for all \( n, m > 0 \).

**Problem 3.** Let \( S \) be a multiplicative subset of a commutative ring \( R \). For any \( R \)-module \( M \), define the localization \( S^{-1}M \) as a module over \( S^{-1}R \). (Hint: Consider an equivalence relation on \( S \times M \).) Show that the correspondence \( M \mapsto S^{-1}M \) extends to a functor \( R\text{-Mod} \to S^{-1}R\text{-Mod} \).

**Problem 4.** Let \( I \) be a (two-sided) ideal of a ring \( R \) and let \( M \) be a (left) \( R \)-module such that \( IM = 0 \). Show that \( M \) has a natural structure of a (left) \( R/I \)-module.

**Problem 5.** Let \( \{M_i\}_{i \in I} \) and \( \{N_j\}_{j \in J} \) be two families of (left) \( R \)-modules. Show that there is a natural isomorphism

\[
\text{Hom}_R \left( \bigoplus_{i \in I} M_i, \prod_{j \in J} N_j \right) \cong \prod_{i \in I, j \in J} \text{Hom}_R(M_i, N_j).
\]

**Problem 6.** Prove that a (left) \( R \)-module over a ring \( R \) with identity generated by one element is isomorphic to \( R/I \) for some (left) ideal \( I \) of \( R \).

**Problem 7.** Let \( F(A) = A_{\text{tors}} \) be the functor from the category of abelian groups to itself (here \( A_{\text{tors}} \) is the subgroup of elements of finite order in \( A \), i.e. torsion elements). Show that \( F \) is left exact.

**Problem 8.** Let

\[
\begin{array}{cccccc}
M_1 & \longrightarrow & M_2 & \longrightarrow & M_3 & \longrightarrow & M_4 & \longrightarrow & M_5 \\
\downarrow f_1 & & \downarrow f_2 & & \downarrow f_3 & & \downarrow f_4 & & \downarrow f_5 \\
N_1 & \longrightarrow & N_2 & \longrightarrow & N_3 & \longrightarrow & N_4 & \longrightarrow & N_5
\end{array}
\]

be a commutative diagram of (left) \( R \)-modules and \( R \)-homomorphisms with exact rows. Prove that

(a) if \( f_1 \) is surjective and \( f_2, f_4 \) are injective, then \( f_3 \) is injective;

(b) if \( f_5 \) is injective and \( f_2, f_4 \) are surjective, then \( f_3 \) is surjective.

**Problem 9.** Let \( M_i \) be a (left) \( R_i \)-module for \( i = 1, \ldots, n \). Show that \( M = M_1 \times \cdots \times M_n \) has a natural structure of a (left) module over the ring \( R = R_1 \times \cdots \times R_n \). Prove that any (left) \( R \)-module is isomorphic to \( M \) as above for some (left) \( R_i \)-modules \( M_i \).

**Problem 10.** Let \( R = \mathbb{Z}[x, y] \). Construct an exact sequence of \( R \)-modules

\[
0 \longrightarrow R \longrightarrow R \oplus R \longrightarrow R \longrightarrow \mathbb{Z} \longrightarrow 0,
\]

where \( f(g(x, y)) = g(0, 0) \). Here \( \mathbb{Z} \) is viewed as an \( R \)-module via \( x \cdot 1 = y \cdot 1 = 0 \).
HOMEWORK 5

Problem 1. (a) Let $I$ be an ideal of a ring $R$. Show that for every left $R$-module $M$, the quotient group $M/IM$ has a natural structure of a left $R/I$-module.

(b) Show that if $B$ is a basis of a free $R$-module $M$, then the image of $B$ in $M/IM$ is a basis of $M/IM$ as a $R/I$-module.

(c) Show that if there exists a surjective ring homomorphism from $R$ to a field, then the rank of a free $R$-module is well-defined.

(d) Prove that every commutative ring $R$ has the property in part (c).

Problem 2. Prove that if every module over a domain $R$ is free, then $R$ is a field.

Problem 3. Show that $\mathbb{Q}$ is not a projective $\mathbb{Z}$-module.

Problem 4. Let $a_1, \ldots, a_n$ be elements of a commutative ring $R$ generating the ideal $R$. Show that the submodule $M$ in $R^n$ consisting of all $n$-tuples $(x_1, \ldots, x_n)$ such that $a_1x_1 + \cdots + a_nx_n = 0$ is projective.

Problem 5. Prove that every $\mathbb{Z}/6\mathbb{Z}$-module is projective and injective. Find a $\mathbb{Z}/4\mathbb{Z}$-module that is neither projective nor injective.

Problem 6. Prove that the ideal $I = (2, 1 + \sqrt{-5})$ in $R = \mathbb{Z}[\sqrt{-5}]$ is a projective $R$-module. Is $I$ a free $R$-module?

Problem 7. Let $S$ be a multiplicative subset in a commutative ring $R$. Show that the localization functor $\text{R-Mod} \to S^{-1}\text{R-Mod}$ is exact.

Problem 8. Prove that if, for a module $M$ over a commutative ring $R$, one has $M_m = 0$ for every maximal ideal $m \subset R$, then $M = 0$. (Here $M_m$ is the localization of $M$ with respect to the multiplicative subset $R \setminus \{p\}$.

Hint: Show that for every non-zero $m \in M$, there is a maximal ideal in $R$ containing all of the elements $a \in R$ such that $am = 0$.

Problem 9. Determine the torsion part of the group $\mathbb{Z}^2/N$, where $N$ is the cyclic subgroup of $\mathbb{Z}^2$ generated by $(6, 21)$.

Problem 10. Let $f : A \to A$ be an endomorphism of an abelian group $A$ such that $f(f(a)) = -a$ for all $a \in A$. Show that there is a structure of a $\mathbb{Z}[i]$-module on $A$ such that $ia = f(a)$ for all $a \in A$. 

HOMEWORK 6

Problem 1. Show that a submodule of a cyclic module over a PID is cyclic.

Problem 2. Let $a$ and $b$ be non-zero elements of a PID $R$. Prove that $R/aR \oplus R/bR \cong R/cR \oplus R/dR$, where $c = \text{lcm}(a, b)$ and $d = \text{gcd}(a, b)$.

Problem 3. Find the invariant factors of $\mathbb{Z}^3/N$, where $N$ is generated by $(-4, 4, 2), (16, -4, -8)$, and $(8, 4, 2)$.

Problem 4. Find the rational canonical form of the linear operator in $\mathbb{R}^3$ given by $\begin{pmatrix} -2 & 0 & 0 \\ -1 & -4 & -1 \\ 2 & 4 & 0 \end{pmatrix}$.

Problem 5. Find the Jordan canonical form of the linear operator in $\mathbb{C}^2$ given by $\begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix}$.

Problem 6. Classify all finitely generated modules over $\mathbb{Z}/n\mathbb{Z}$.

Problem 7. Classify all finitely generated modules over $S^{-1}\mathbb{Z}$ for each of the cases $S = \{p^n \mid n \geq 0\}$ and $S = \mathbb{Z}/p\mathbb{Z}$. (Here $p$ is a prime integer.)

Problem 8. Let $N$ be a submodule of a finitely generated free module $F$ over a PID $R$. Show that $N$ is a direct summand of $F$ if and only if $N \cap aF = aN$ for all $a \in R$.

Problem 9. Let $A$ be a nilpotent $n \times n$ matrix. Show that the invariant factors of $A$ are powers of $x$. Prove that $A^n = 0$.

Problem 10. Prove that an $n \times n$ matrix is similar to its transpose.