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HOMEWORK A1

Problem 1. Let \(a_1, a_2, \ldots, a_n\) be elements of a group \(G\). Define the product of the \(a_i\)'s by induction:
\[
a_1a_2 \cdots a_n = (a_1a_2 \cdots a_{n-1})a_n.
\]
(a) Prove that
\[
a_1a_2 \cdots a_n b_1b_2 \cdots b_m = (a_1a_2 \cdots a_n)(b_1b_2 \cdots b_m).
\]
(b) Prove that \(a_1a_2 \cdots a_n\) is equal to the product of the \(a_i\)’s with parentheses inserted arbitrarily.

Problem 2. (a) Prove that for any natural \(n\), the set of all complex \(n\)-th roots of unity forms a group with respect to the complex multiplication. Show that this group is cyclic.
(b) Prove that if \(G\) is a cyclic group of order \(n\) and \(k\) divides \(n\), then \(G\) has exactly one subgroup of order \(k\).

Problem 3. (a) Show that if \(K\) and \(N\) are two finite subgroups in \(G\) of relatively prime orders, then \(K \cap N = 1\).
(b) Show that if a group \(G\) has only a finite number of subgroups, then \(G\) is finite.

Problem 4. (a) Let \(G\) be a group of order \(n\). Show that \(a^n = 1\) for all \(a \in G\).
(b) Prove that a group \(G\) is cyclic if and only if there is an element \(a \in G\) with \(\text{ord}(a) = |G|\).
(c) Show that every group of prime order is cyclic.

Problem 5. (a) Show that if \(a^2 = 1\) for all elements \(a\) of a group \(G\), then \(G\) is abelian.
(b) Prove that if \(G\) is a finite group of even order, then \(G\) contains an element \(a\) such that \(a^2 = 1\) and \(a \neq 1\).
(c) Show that every subgroup of index \(2\) is normal.

Problem 6. Find all groups (up to isomorphism) of order \(\leq 5\). What is the smallest order of a non-cyclic group?

Problem 7. Find a non-normal subgroup in the symmetric group \(S_3\).

Problem 8. Let \(n\) be a natural number. Show that the map
\[
f : \mathbb{Q}/\mathbb{Z} \to \mathbb{Q}/\mathbb{Z}, \quad f(a + \mathbb{Z}) = na + \mathbb{Z}
\]
is a well-defined homomorphism. Find \(\ker f\) and \(\text{im} f\).

Problem 9. (a) Show that \(\mathbb{Z}/6\mathbb{Z} \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}\).
(b) Prove that \(n\mathbb{Z}/mn\mathbb{Z} \cong \mathbb{Z}/m\mathbb{Z}\).

Problem 10. Let \(f : G \to H\) be a surjective group homomorphism and let \(H'\) be a normal subgroup in \(H\). Show that \(G' = f^{-1}(H')\) is a normal subgroup in \(G\) and \(G/G' \cong H/H'\).
Problem 1. Let $G$ be a group and $a, b \in G$.

(a) Prove that $a^n \cdot a^m = a^{n+m}$ and $(a^n)^m = a^{nm}$.

(b) Prove that $\text{ord}(a^n) = \text{ord} a / \gcd(n, \text{ord} a)$ if $\text{ord} a < \infty$.

(c) Prove that $\text{ord}(ab) = \text{ord} a \cdot \text{ord} b$ if $a$ and $b$ commute and $\gcd(\text{ord} a, \text{ord} b) = 1$.

Problem 2. Let $H \leq G$ be a subgroup. Show that the correspondence $Ha \mapsto (Ha)^{-1} = a^{-1}H$ is a bijection between the sets of right and left cosets.

Problem 3. Let $H \leq G$ be a subgroup. Suppose that for any $a \in G$, there exists $b \in G$ such that $aH = Hb$. Show that $H$ is normal in $G$.

Problem 4. Let $f : G \to H$ be a surjective group homomorphism.

(a) Let $H'$ be a subgroup of $H$. Show that $G' = f^{-1}(H')$ is a subgroup of $G$. Prove that the correspondence $H' \mapsto G'$ is a bijection between the set of all subgroups of $H$ and the set of all subgroups of $G$ containing $\ker f$.

(b) Let $H'$ be a normal subgroup of $H$. Show that $G' = f^{-1}(H')$ is a normal subgroup of $G$. Prove that $G/G' \cong H/H'$ and the correspondence $H' \mapsto G'$ is a bijection between the set of all normal subgroups of $H$ and the set of all normal subgroups of $G$ containing $\ker f$.

Problem 5. (a) Let $N$ be a subgroup in the center $Z(G)$ of $G$. Show that $N$ is normal in $G$. Prove that if the factor group $G/N$ is cyclic, then $G$ is abelian.

(b) Prove that every group of order $p^2$ (for a prime $p$) is abelian.

Problem 6. Prove that if a group $G$ contains a subgroup $H$ of finite index, then $G$ contains a normal subgroup $N$ of finite index such that $N \subseteq H$.

*Hint: Consider the homomorphism of $G$ to the symmetric group of all left cosets of $H$ in $G$ taking any $x \in G$ to $f_x$ defined by $f_x(aH) = xaH$.*

Problem 7. (a) Show that the group $\text{Inn} G$ of all inner automorphisms of a group $G$ (given by $a \mapsto gag^{-1}$ for some $g \in G$) is a normal subgroup in $\text{Aut} G$.

(b) Find all automorphisms of all (finite and infinite) cyclic groups.

Problem 8. Prove that if $G$ has no non-trivial automorphisms, then $G$ is abelian and $g^2 = 1$ for all $g \in G$.

Problem 9. Let $x$ and $x'$ be two elements in the same orbit under some action of a group $G$ on a set. Show that the stabilisers $G_x$ and $G_{x'}$ are conjugate in $G$.

Problem 10. Let a group $G$ act on two sets $X$ and $Y$. We say that $X$ and $Y$ are $G$-isomorphic if there is a bijection $f : X \to Y$ such that $f(gx) = g(f(x))$ for every $x \in X$ and $g \in G$. Prove that if $G$ acts on $X$ transitively, then $X$ is $G$-isomorphic to the set of left cosets $G/H$ for some subgroup $H \leq G$ (with the action of $G$ on $G/H$ by left translations).
HOMEWORK A3

Problem 1. Let $H$ be a $p$-subgroup of a finite group $G$. Show that if $H$ is not a Sylow $p$-subgroup, then $N_G(H) \neq H$.

Problem 2. Let $G$ be a $p$-group and let $k$ be a divisor of $|G|$. Prove that $G$ contains a normal subgroup of order $k$.

Problem 3. Prove that if a group $G$ contains a subgroup $H$ of finite index, then $G$ contains a normal subgroup $N$ of finite index such that $N \subset H$.

Hint: Consider the action of $G$ on $G/H$ by left translations.

Problem 4. Let $G$ be a $p$-group and $H$ a normal subgroup in $G$ of order $p$. Show that $H \subset Z(G)$.

Hint: Consider the action of $G$ on $H$ by conjugation.

Problem 5. (a) A subgroup $H$ of $G$ is called characteristic if $f(H) = H$ for every automorphism $f$ of $G$. Show that a characteristic subgroup $H$ is normal in $G$.

(b) Prove that if $K$ is a characteristic subgroup of $H$ and $H$ is a characteristic subgroup of $G$, then $K$ is characteristic in $G$.

Problem 6. For a group $G$, set $G^{(0)} = G$ and $G^{(i+1)} = [G^{(i)}, G^{(i)}]$ for $i \geq 0$. Show that $G^{(i)}$ is a characteristic subgroup of $G$.

Problem 7. (a) For any two subgroups $K$ and $H$ of a group $G$, denote by $[K, H]$ the subgroup in $G$ generated by the commutators $[k, h] = khk^{-1}h^{-1}$ for all $k \in K$ and $h \in H$. Show that if $K$ and $H$ are normal in $G$, then so is $[K, H]$.

(b) Prove that $[G, H]$ is normal in $G$ for every subgroup $H \leq G$.

Problem 8. Let $G$ be a group and $Z(G)$ the center of group. Show that if $G/Z(G)$ is nilpotent, then so is $G$.

Problem 9. Assume that a subset $S \subset G$ of a group satisfies $gSg^{-1} \subset S$ for all $g \in G$. Prove that the subgroup generated by $S$ is normal in $G$.

Problem 10. Let $N$ be an abelian normal subgroup in a finite group $G$. Assume that the orders $|G/N|$ and $|\text{Aut } N|$ are relatively prime. Prove that $N$ is contained in the center of $G$. 
HOMEWORK A4

Problem 1. Determine all conjugacy classes in $S_n$ for $n \leq 4$.

Problem 2. Determine all subgroups in $A_4$. Show that $A_4$ has no subgroups of order 6.

Problem 3. (a) Prove that $S_n$ is generated by $(1, 2), (1, 3), \ldots, (1, n)$.

(b) Prove that $S_n$ is generated by two cycles $(1, 2)$ and $(1, 2, \ldots, n)$.

Problem 4. Show that $A_n$ ($n \geq 4$) and $S_n$ ($n \geq 3$) have trivial center.

Problem 5. (a) Show that the centralizer of $A_n$ in $S_n$ (the subgroup in $S_n$ consisting of all elements which commute with all elements of $A_n$) is trivial if $n \geq 4$.

(b) Let $g \in S_n$ be an odd transformation. Show that the map $f : A_n \to A_n$ given by $f(x) = gxg^{-1}$ is an automorphism. Prove that $f$ is not an inner automorphism if $n \geq 3$.

Problem 6. Prove that every automorphism of $S_3$ is inner and that $\text{Aut} S_3$ is isomorphic to $S_3$.

Problem 7. Describe all Sylow subgroups of $S_5$.

Problem 8. Show that every subgroup in $S_n$ of index $n$ is isomorphic to $S_{n-1}$.

Hint: For a subgroup $H \leq S_n$ of index $n$, consider the homomorphism $S_n \to S(S_n/H)$ induced by the action of $S_n$ on $S_n/H$ by left translations.

Problem 9. (a) Show that for $n \neq 4$, every proper subgroup in $A_n$ has index at least $n$.

Hint: Use the hint to Problem 8.

(b) Prove that there are no injective homomorphisms $S_n \to A_{n+1}$ for $n \geq 2$.

Problem 10. (a) Show that for any $n \geq 1$, there is an injective homomorphism $S_n \to A_{n+2}$.

(b) Prove that every finite group is isomorphic to a subgroup of a finite simple group.
HOMEWORK A5

Problem 1. Prove that two elements $\sigma$ and $\tau$ in $S_n$ are conjugate if and only if $\text{type}(\sigma) = \text{type}(\tau)$.

Problem 2. Prove that $S_4$ acts by conjugation on the set of all non-trivial elements of the normal subgroup $N = \{\text{id}, (12)(34), (13)(24), (14)(23)\}$. Using this action, prove that $S_4/N$ is isomorphic to $S_3$.

Problem 3. Let $\sigma = (12\cdots n) \in S_n$. Show that the conjugacy class of $\sigma$ has $(n-1)!$ elements. Show that the centralizer of $\sigma$ is the cyclic subgroup generated by $\sigma$.

Problem 4. Prove the following useful counting result: Let $H$ be a subgroup of a finite group $G$ with $H \neq G$. Suppose that $|G|$ does not divide $[G : H]$!. Then $G$ contains a proper normal subgroup $N$ such that $N$ is a subgroup of $H$. In particular, $G$ is not simple.

Problem 5. Prove that all groups of order $2p^n$ and $4p^n$ are not simple (where $p$ is a prime number).

Problem 6. (a) Let $H \leq G$ be a subgroup. Prove that if $H$ is contained in the center of $G$ and the factor group $G/H$ is cyclic, then $G$ is abelian.

(b) Prove that any group of order $p^2$ is abelian (where $p$ is a prime integer).

Problem 7. Let $G$ be a non-abelian group of order $p^3$ (where $p$ is a prime integer). Prove that the center $Z(G)$ of $G$ coincides with the commutator subgroup $G' = [G, G]$.

Problem 8. Let $G$ be a semidirect product of a cyclic normal subgroup $N$ of order $n$ and an abelian group $K$. Show that if $|K|$ is relatively prime to $\varphi(n)$ (where $\varphi$ is the Euler function), then $G$ is abelian.

Problem 9. Determine the center of the dihedral group $D_n$.

Problem 10. Show that the exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Q} \longrightarrow \mathbb{Q}/\mathbb{Z} \longrightarrow 0$$

is not split.
HOMEWORK A6

Problem 1. For every two non-zero integers \( n \) and \( m \), construct an exact sequence

\[
0 \longrightarrow \mathbb{Z}/n\mathbb{Z} \longrightarrow \mathbb{Z}/nm\mathbb{Z} \longrightarrow \mathbb{Z}/m\mathbb{Z} \longrightarrow 0.
\]

For which \( n \) and \( m \) is the sequence split?

Problem 2. Let \( N \) be the normal subgroup in \( G \ast H \) generated by \( G \subset G \ast H \). Prove that \( (G \ast H)/N \cong H \).

Problem 3. Let \( D_\infty = \mathbb{Z} \rtimes (\mathbb{Z}/2\mathbb{Z}) \) with respect to the unique isomorphism \( \mathbb{Z}/2\mathbb{Z} \to \text{Aut} \mathbb{Z} \). Prove that \( D_\infty \cong (\mathbb{Z}/2\mathbb{Z}) \ast (\mathbb{Z}/2\mathbb{Z}) \).

Problem 4. Show that there exists a surjective homomorphism \((\mathbb{Z}/n\mathbb{Z}) \ast (\mathbb{Z}/2\mathbb{Z}) \to S_n\).

Problem 5. Prove that the group \( SL_2(\mathbb{Z}) \) is generated by the two matrices

\[
\begin{pmatrix}
0 & -1 \\
1 & 0
\end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
0 & -1 \\
1 & 1
\end{pmatrix}.
\]

Problem 6. Let \( H \) and \( K \) be two subgroups in \( G \). Assume that \( G \) acts on a set \( X \) and that there are two subsets \( A, B \subset X \) and an element \( x \in X \backslash (A \cup B) \) such that \( h(A \cup \{x\}) \subset B \) for every \( h \in H, h \neq 1 \), and \( k(B \cup \{x\}) \subset A \) for every \( k \in K, k \neq 1 \). Prove that the natural homomorphism \( H \ast K \to G \) is injective.

Problem 7. Let \( G \) and \( H \) be two non-trivial groups. Show that \( G \ast H \) is an infinite group with trivial center.

Problem 8. Let \( X \) be a subset in a group \( G \). Prove that \( \langle \langle X \rangle \rangle = \langle Y \rangle \), where \( Y = \bigcup_{g \in G} gXg^{-1} \).

Problem 9. Let \( G \) be the group defined by the generators \( a \) and \( b \) and relations \( w^3 = 1 \) for all words \( w \) in \( a \) and \( b \). Show that \( G \) is finite, and find \( |G| \).

Problem 10. Prove that if the free groups \( F(X) \) and \( F(Y) \) for the finite sets \( X \) and \( Y \) are isomorphic, then \( |X| = |Y| \).
HOMEWORK A7

Problem 1. Show that $Q_8 = \langle x, y, c \mid c^2, x^2 c, y^2 c, xyxc \rangle$.

Problem 2. Prove that if $X_1$ and $X_2$ are disjoint sets, then
\[ \langle X_1 \mid R_1 \rangle \ast \langle X_2 \mid R_2 \rangle \cong \langle X_1 \cup X_2 \mid R_1 \cup R_2 \rangle. \]

Problem 3. Prove that if $X$ is a set of $n$ elements, then $F(X) \cong \mathbb{Z} \ast \cdots \ast \mathbb{Z}$ ($n$ times).

Problem 4. Prove that the natural homomorphisms $G \to G \ast H$ and $H \to G \ast H$ are injective.

Problem 5. Let $f$ be a homomorphism from the free group on $x$ and $y$ to the cyclic group $\{\pm 1\}$ taking $x$ and $y$ to $-1$. Prove that the kernel of $f$ is a free group freely generated by $x^2, y^2,$ and $xy$.

Problem 6. Give an example of a category $C$ and a full subcategory $C'$ such that the initial objects of $C$ and $C'$ are different.

Problem 7. (a) A morphism $f : A \to B$ in a category $C$ is called a monomorphism if for any two morphisms $g, h : C \to A$, the equality $fg = fh$ implies $g = h$. Show that the composition of two monomorphisms is a monomorphism. Determine monomorphisms in Set and Grp.

(b) Define the dual notion of epimorphism and solve the dual problems in Set and Grp.

Problem 8. (a) An object $P$ of $C$ is called projective if for every epimorphism $f : B \to C$ and every morphism $g : P \to C$, there exists a morphism $h : P \to B$ such that $fh = g$. Show that free groups in Grp are projective objects.

(b) Define the dual notion of injective objects. Show that in the category of finite-dimensional vector spaces over a given field, every object is projective and injective.

Problem 9. (a) Let $X$ be an object of a category $C$. Consider a new category $C/X$ with objects the morphisms $f : Y \to X$ for $Y$ in $C$ and morphisms between $f : Y \to X$ and $g : Z \to X$ being morphisms $h : Y \to Z$ such that $gh = f$. The product of two objects $f : Y \to X$ and $g : Z \to X$ in $C/X$ is called the fiber product of $Y$ and $Z$ over $X$, denoted $Y \times_X Z$. Show that fiber products exist when $C/X$ is Set$/X$, Grp$/X$, or Ab$/X$ for any object $X$ in $C$.

(b) Define the dual notion and solve the dual problems.

Problem 10. (a) The kernel of a pair of morphisms $f, g : X \to Y$ in $C$ is a morphism $h : Z \to X$ such that $fh = gh$ and for any morphism $i : T \to X$ in $C$ such that $fi = gi$, there exists a unique morphism $j : T \to Z$ such that $hj = i$. Show that kernels exist in Set and Ab.

(b) Define the dual notion of a cokernel. Show that cokernels exist in Set and Ab.
**HOMEWORK A8**

**Problem 1.** Let $X$ be a final object of a category $C$. Prove that $X \times Y \cong Y$ for every $Y \in \text{Ob } C$.

**Problem 2.** Let $F : C^{\text{op}} \to \text{Grp}$ be a functor such that the composition of $F$ with the forgetful functor $\text{Grp} \to \text{Set}$ is corepresented by an object $X$. Prove that $X$ has the structure of a group object in $C$.

**Problem 3.** Let $F : C \to \text{Set}$ be a functor. Consider a new category $D$ with objects the pairs $(X, u)$, where $X$ is an object in $C$ and $u \in F(X)$. A morphism between $(X, u)$ and $(X', u')$ in $D$ is a morphism $f : X \to X'$ in $C$ such that $F(f)(u) = u'$. Prove that if $(X, u)$ is an initial object in $D$, then the functor $F$ is represented by $X$.

**Problem 4.** Prove that the category $\text{Set}^{\text{op}}$ is not equivalent to $\text{Set}$.

**Problem 5.** Construct a functor $F : \text{Grp} \to \text{Set}$ that assigns to each group the set of all its subgroups.

**Problem 6.** Prove that if a functor $F : C \to D$ is an equivalence of categories, then $X$ and $Y$ are isomorphic in $C$ if and only if $F(X)$ and $F(Y)$ are isomorphic in $D$.

**Problem 7.** Consider a category $C$ with the set of objects $\{0, 1, 2, \ldots\}$ and morphisms $\text{Hom}(i, j)$ the set of all $j \times i$ matrices over $\mathbb{R}$ (with composition the matrix multiplication). Construct an equivalence between $C$ and the category of real finite-dimensional vector spaces.

**Problem 8.** Let $F : A \to B$ be a functor. Suppose that $F$ has a left adjoint functor. Prove that $F$ takes terminal objects of $A$ to terminal objects of $B$.

**Problem 9.** Determine the limit and colimit of the diagram

$$X \begin{array}{c} \longrightarrow \end{array} Y$$

in $\text{Set}$.

**Problem 10.** Prove that limits and colimits exist in $\text{Set}$ and $\text{Grp}$.
HOMEWORK A9

Problem 1. Let \((G, m, e, i)\) be a group object in the category of groups. Prove that \(G\) is an abelian group and \(m : G \times G \to G\), \(e : 1 \to G\), and \(i : G \to G\) are the product map, the unit map, and the inverse map, respectively.

Problem 2. Let \(F : \mathcal{I} \to \mathcal{C}\) be a functor from a small category \(\mathcal{I}\) with an initial object \(i\). Prove that \(\text{lim } F \cong F(i)\).

Problem 3. Determine initial and terminal objects in the category of rings.

Problem 4. Prove that a finite non-zero ring with no zero divisors is a division ring and a finite integral domain is a field.

Problem 5. Let \(R\) be the set of all \(2 \times 2\) matrices over \(\mathbb{C}\) of the form
\[
\begin{pmatrix}
u & v \\
-v & u
\end{pmatrix}.
\]
Show that \(R\) is a subring with identity in \(\text{Mat}_2(\mathbb{C})\) isomorphic to the ring of real quaternions \(\mathbb{H}\).

Problem 6. (a) Prove that a non-zero matrix \(a \in \text{Mat}_n(F)\), where \(F\) is a field, is a zero divisor if and only if \(\det a = 0\).

(b) Prove that a non-zero matrix \(a \in \text{Mat}_n(R)\), where \(R\) is a commutative ring, is a zero divisor if \(\det a = 0\).

Problem 7. Let \(S = \text{Mat}_n(R)\), where \(R\) is a ring with identity. Show that for any ideal \(J \subset S\), there is a unique ideal \(I \subset R\) such that \(J\) is the set of all \(n \times n\) matrices with the elements in \(I\).

Problem 8. (a) Let \(f : R \to S\) be a ring homomorphism, \(I\) be an ideal in \(R\), and \(J\) be an ideal in \(S\). Show that \(f^{-1}(J)\) is an ideal in \(R\) that contains \(\ker f\).

(b) Show that if \(f\) is surjective, then \(f(I)\) is an ideal in \(S\), but if \(f\) is not surjective, then \(f(I)\) need not be an ideal in \(S\).

Problem 9. (a) An element \(a\) of a ring \(R\) is called nilpotent if \(a^n = 0\) for some \(n\). Show that if \(R\) is a commutative ring, then the set \(\text{Nil } R\) of all nilpotent elements in \(R\) is an ideal, called the nilradical of \(R\). Prove that the factor ring \(R/\text{Nil } R\) has no non-zero nilpotent elements.

(b) Prove that a polynomial \(f(x) = a_0 + a_1 x + \cdots + a_n x^n \in R[x]\) (where \(R\) is a commutative ring) is nilpotent if and only if all \(a_i\) are nilpotent in \(R\).

Problem 10. (a) Prove that if \(a\) is a nilpotent element of a ring \(R\), then \(1 + a\) is invertible.

(b) Prove that a polynomial \(f(x) = a_0 + a_1 x + \cdots + a_n x^n \in R[x]\) (where \(R\) is a commutative ring) is invertible in \(R[x]\) if and only if \(a_0\) is invertible and \(a_i\) is nilpotent in \(R\) for \(i \geq 1\).