

Math 33b Midterm solutions

1) Solve the initial value problem

$$y + xy \cot x + x \frac{dy}{dx} = 0$$

$$y\left(\frac{\pi}{2}\right) = \frac{2}{\pi}.$$

It may be useful to consider the function $u(x) = \sin x$.

Solution 1: multiply both sides of the differential equation by $\sin x$ to get

$$y \sin x + xy \cos x + x \sin x \frac{dy}{dx} = 0,$$

which is exact. So we need to find $F(x, y)$ such that $\frac{\partial F}{\partial x} = y \sin x + xy \cos x$ and $\frac{\partial F}{\partial y} = x \sin x$. The second equation seems easier to work with, so we start there....

$$\begin{aligned} F(x, y) &= \int x \sin x \, dy + \phi(x) \\ &= yx \sin x + \phi(x). \end{aligned}$$

Using $\frac{\partial F}{\partial x} = y \sin x + xy \cos x$, we get

$$yx \cos x + y \sin x + \phi'(x) = y \sin x + xy \cos x,$$

and so $\phi = 0$, and $F(x, y) = yx \sin x$. The solution to the I.V.P. will then be $yx \sin x = C$ for some C . Subbing in $\frac{\pi}{2}$ for x and $\frac{2}{\pi}$ for y , we get $C = 1$, and the final solution is $yx \sin x = 1$.

Solution 2: the equation is seperable. Rearranging it gives

$$y + xy \cot x + x \frac{dy}{dx} = 0$$

$$\begin{aligned} \frac{dy}{dx} &= \frac{-1 - x \cot x}{x} \\ &= \frac{-1}{x} - \cot x. \end{aligned}$$

Integrating both sides of the equation (use the substitution $u = \sin x$ to integrate \cot), we get

$$\ln |y| = -\ln |x| - \ln |\sin x| + C.$$

As before, using the initial condition gives $C = 1$, and determines the branch of the logarithms, giving

$$y = \frac{1}{x \sin x}.$$

2) Find a general solution to the equation

$$y' = \frac{y}{t} + ty^2.$$

The substitution $z = 1/y$ may be useful to you.

Solution: we use the substitution. $y = 1/z$, so

$$y' = \frac{-z'}{z^2},$$

and the equation becomes

$$\frac{-z'}{z^2} = \frac{1}{tz} + \frac{t}{z^2}.$$

Rearranging, we get the linear equation

$$z' + \frac{z}{t} = -t.$$

We multiply both sides by the integrating factor t to obtain

$$(tz)' = -t^2,$$

and so

$$z = \frac{-t^3 + C}{3t},$$

and

$$y = \frac{3t}{-t^3 + C}.$$

3) Find the general solution to the equation

$$y'' + y' + 2y = \frac{13}{2} \sin 2t$$

The characteristic polynomial of $y'' + y' + 2y$ is $x^2 + x + 2$, with roots

$$\lambda_1 = \frac{-1}{2} + \frac{i\sqrt{7}}{2}$$

and

$$\lambda_2 = \frac{-1}{2} - \frac{i\sqrt{7}}{2}.$$

Thus the general solution to the homogenous equation $y'' + y' + 2y = 0$ is

$$e^{\frac{-t}{2}} \left(C_1 \cos \frac{\sqrt{7}}{2} t + C_2 \sin \frac{\sqrt{7}}{2} t \right).$$

Now we find a particular solution to the inhomogenous equation. Guessing a solution of the form $a \cos 2t + b \sin 2t$, and plugging into the equation gives

$$-4a \cos 2t - 4b \sin 2t - 2a \sin 2t + 2b \cos 2t + 2a \cos 2t + 2b \sin 2t = \frac{13}{2} \sin 2t,$$

which leads to

$$\begin{aligned} -4a + 2b + 2a &= 0 \\ -4b - 2a + 2b &= \frac{13}{2}, \end{aligned}$$

and so $a = b = \frac{-13}{8}$, and the general solution to the homogenous equation is

$$e^{\frac{-t}{2}} \left(C_1 \cos \frac{\sqrt{7}}{2} t + C_2 \sin \frac{\sqrt{7}}{2} t \right) - \frac{13}{8} \cos 2t - \frac{13}{8} \sin 2t.$$

4) Show that if y is a solution to

$$y' = \sin y^2 \cos ty$$

with $y(0) = \pi/2$ then $0 < y(t) < \pi$ for all t .

Solution: first notice that $y = 0$ and $y = \pi$ are steady state solutions to the equation. Next, we check that the equations satisfies the conditions of the existence and uniqueness theorems:

$$f(t, y) = \sin y^2 \cos ty$$

and

$$\frac{\partial f}{\partial y} = 2 \cos y \sin y \cos ty - t \sin y^2 \sin ty$$

are continuous on the whole t, y plane. In particular, if $y(t) = 0$ for any solution y and any t , then by uniqueness, $y(t) = 0$ for all t , and similarly with $y(t) = \pi$. If $y(0) = \pi/2$, then if y is ever greater than π , by the intermediate value theorem, there was some point t where $y(t)$ was equal to π , which is a contradiction. Similarly we get a contradiction if y is ever less than 0.