Quiz 1 solutions

i) Show that the vector field

$$\begin{pmatrix} y/x \\ \ln x + 1 \end{pmatrix}$$

is exact.

Solution: we simply need to check that

$$\frac{\partial}{\partial y} y/x = \frac{\partial}{\partial x} (\ln x + 1) \, .$$

Both are equal to $1/x$.

ii) Find a solution to the initial value problem

$$\frac{y}{x} + (\ln x + 1)\frac{dy}{dx} = 0,$$

$$y(1) = 1.$$

Solution: there are (at least) two ways of solving this. Method 1: We know that this equation is exact, so let

$$F(x, y) = \int \frac{y}{x} \, dx + \phi(y) = y \ln x + \phi(y),$$

with ϕ to be determined so that

$$\frac{\partial F}{\partial y} = \ln x + 1.$$

This gives

$$\ln x + \phi'(y) = \ln x + 1,$$

and so $\phi(y) = y$. The general solution to the equation is then

$$F(x, y) = y \ln x + y = C.$$

To find C for the given initial conditions plug in 1 for x and 1 for y:

$$1 \ln 1 + 1 = 1 = C.$$

So the final answer is

$$y \ln x + y = 1.$$

Method 2: This equation is seperable. Separating variables gives:

$$\frac{\frac{dy}{dx}}{y} = -\frac{1}{x(\ln x + 1)}.$$

The left side is easy to integrate with respect to x, giving $\ln y$. The right side can be integrated with a substitution: let $u = \ln x + 1$. Then $u' = 1/x$, and the right hand side transforms to u'/u, and so

$$-\int \frac{1}{x(\ln x + 1)}\, dx = -\ln|\ln x + 1| + C.$$

Exponentiating both sides gives

$$|y| = \frac{e^C}{|\ln x + 1|} = \frac{K}{|\ln x + 1|}.$$

Plugging 1 in for y and x gives $K = 1$, and the choice of the positive branch of the absolute value (or equivalently, $K = -1$ and the negative branch of the absolute value), and the final solution is

$$y = \frac{1}{\ln x + 1}.$$