

Imagine we have n points arranged on a circle, and a particle hops around the nodes. Suppose at time t the particle is at node i . We assume the probability that in the time interval $[t, t + h)$ the particle moves to its neighbor $i + 1$ is h , and the probability that it moves to its neighbor $i - 1$ is h , and the probability that it does not move is $1 - 2h$. Two notes... first, if $i = n$, $i + 1 = 1$; that is, $n \equiv 0$. Second: by time $h = 1/2$, the particle has surely moved, so the clock has surely reset, and $1 - 2h$ is not negative.

We will derive a differential equation for the function $p_j(t)$, that is, the probability that the particle is at node j at time t . By definition,

$$\frac{\partial p_j}{\partial t} = \lim_{h \rightarrow 0} \frac{p_j(t+h) - p_j(t)}{h}.$$

We would like to derive the formula

$$p_j(t+h) = hp_{j+1}(t) + hp_{j-1}(t) + (1-2h)p_j(t) + o(h), \quad (1)$$

where, as usual, $o(h)$ just means a function of h such that $\lim_{h \rightarrow 0} o(h)/h = 0$. If that is true,

$$\begin{aligned} & \frac{p_j(t+h) - p_j(t)}{h} \\ &= \frac{hp_{j+1}(t) + hp_{j-1}(t) + (1-2h)p_j(t) + o(h) - p_j(t)}{h} \\ &= \frac{h[p_{j+1}(t) + p_{j-1}(t) - 2p_j(t)] + o(h)}{h}, \end{aligned}$$

so

$$\boxed{\frac{\partial p_j}{\partial t} = p_{j+1}(t) + p_{j-1}(t) - 2p_j(t).} \quad (2)$$

To show (1), let's look more carefully at $p_j(t+h)$, the probability that at time $t+h$ the particle is at j . We can break this up into the history of the particle from time t ... By γ denote a path on the points; that is, γ is just a sequence of numbers in $\{1, \dots, n\}$. Let $|\gamma|$ be the length of the path, and let $\gamma(k)$ be the k th number in the sequence. Then

$$p_j(t+h) = \sum_{\gamma(\text{end})=j} P(\gamma),$$

where $P(\gamma)$ is the probability that the particle took the path γ in the time interval $[t, t+h)$. What does the formula say in english? The probability of a certain present (at time $t+h$) is the sum of the probabilities of all the histories from t to $t+h$ which end up with the system as it is in the present.

In this case, the histories are just the paths, so the probability that we are at j at time $t+h$ is the sum over m of all the paths length of length m which end up at j , weighted by the probability of taking that particular path. We now split the above sum over paths into two pieces:

$$p_j(t+h) = \sum_{\gamma(\text{end})=j, |\gamma|\leq 1} P(\gamma) + \sum_{\gamma(\text{end})=j, |\gamma|\geq 2} P(\gamma).$$

Denote by $g(h)$ the probability that the particle took one of the long paths; that is,

$$g(h) = \sum_{\gamma(\text{end})=j, |\gamma|\geq 2} P(\gamma).$$

We now just need to show that $g(h)$ goes to zero faster than h and we will have (1). This is because the only paths of length 1 ending up at j are the ones from $j-1$ and $j+1$, and the probability that those paths were taken is $hp_{j-1}(t)$ and $hp_{j+1}(t)$, respectively, and there is exactly one path of length 0 ending in j , which was taken with probability $(1-2h)p_j(t)$; thus

$$\sum_{\gamma(m)=j, |\gamma|\leq 1} P(\gamma) = hp_{j+1}(t) + hp_{j-1}(t) + (1-2h)p_j(t).$$

To show $g(h)$ goes to zero faster than h , note that the total number of paths of length m ending up at particle j is 2^m . On the other hand, the probability of making m specified jumps in the time interval $[t, t+h]$ is less than h^m , because each jump has probability less than h , and the probabilities are independent. So the total probability of taking some path of length m to get to j in the time interval $[t, t+h]$ is less than $2^m h^m$, and

$$\begin{aligned} |g(h)| &\leq [(2h)^2 + (2h)^3 + \dots] \\ &= (2h)^2 [1 + 2h + (2h)^2 + \dots], \end{aligned}$$

and if $h < 1/2$ (which it will be, because we are taking the limit as $h \mapsto 0$,

$$= (2h)^2 \frac{1}{1-2h},$$

and so $\lim_{h \rightarrow 0} g(h)/h = 0$. We get (1), and thus boxed equation (2).

Now what? We now have the n dimensional linear equation

$$\dot{p} = Ap,$$

where

$$A = \begin{pmatrix} -2 & 1 & 0 & \cdots & 0 & 1 \\ 1 & -2 & 1 & 0 & \cdots & 0 \\ 0 & 1 & -2 & 1 & & \vdots \\ \vdots & & \ddots & & & 0 \\ 0 & & & & & 1 \\ 1 & 0 & \cdots & 0 & 1 & -2 \end{pmatrix}.$$

If we can find eigenfunctions of A , we can solve the equation... As usual, looking at the symmetry of the problem gives the answer. If we cycle the indices, e.g. $j \mapsto j + 1$, $j + 1 \mapsto j + 2$, etc., A remains unchanged. Thus the eigenspaces of A must remain unchanged when we cycle the indices. Let us hope for a moment that all the eigenspaces are 1- d and we have n distinct eigenvectors. If this is the case, then each eigenvector v has the property that if you cycle the indices by 1, you multiply v by a constant a ; because we assumed the eigenspace for v is 1- d , so any other vector in the same eigenspace is a constant multiple of v . In symbols

$$\begin{pmatrix} v_n \\ v_1 \\ v_2 \\ \vdots \\ v_{n-1} \end{pmatrix} = a \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_{n-1} \\ v_n \end{pmatrix}.$$

If we repeat,

$$\begin{pmatrix} v_{n-1} \\ v_n \\ v_1 \\ \vdots \\ v_{n-2} \end{pmatrix} = a^2 \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_{n-1} \\ v_n \end{pmatrix},$$

and cycling all the way around gives

$$\begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_{n-1} \\ v_n \end{pmatrix} = a^n \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_{n-1} \\ v_n \end{pmatrix},$$

and we find $a^n = 1$. So $a = e^{i2\pi k/n}$ for some k in $\{1, \dots, n\}$. And there is more- by looking at each coordinate, we find $av_j = v_{j+1}$, which if we

repeat gives $v_j = a^{j-1}v_1$. Picking $v_1 = 1$ for convenience, our guesses at the eigenvectors have to be the vectors

$$\begin{pmatrix} 1 \\ a \\ a^2 \\ \vdots \\ a^{n-1} \end{pmatrix}$$

where $a = e^{i2\pi k/n}$ for some k in $\{1, \dots, n\}$, that is, we guess $v = v_k$ is of the form

$$\begin{pmatrix} 1 \\ e^{i2\pi k/n} \\ e^{i2\pi 2k/n} \\ \vdots \\ e^{i2\pi(n-1)k/n} \end{pmatrix}.$$

Is our guess right?

$$\begin{aligned} (Av_k)_j &= e^{i2\pi(j-1)k/n} + e^{i2\pi(j+1)k/n} - 2e^{i2\pi jk/n} \\ &= e^{i2\pi jk/n} [e^{-i2\pi k/n} + e^{i2\pi k/n} - 2] \\ &= 2(\cos 2\pi k/n - 1)e^{i2\pi jk/n}. \end{aligned}$$

So in fact the guess is right, and the k th eigenvector has eigenvalue $2(\cos 2\pi k/n - 1)$.

Something is slightly amiss, though: since $n = 0$, each pair $k = -(n - k) = k - n$. In other words, going around the circle forwards k is the same as going backwards $n - k$. But since $\cos -\theta = \cos \theta$, this means that our eigenspaces were not 1-dimensional. So we were lucky in our guesses... This means that each pair v_k and v_{n-k} has the same eigenvalue; and in fact if we look closely at v_k and v_{n-k} , we realize they are conjugates: $v_k = \overline{v_{n-k}}$. Thus we have real eigenvectors w_k of the form

$$\begin{pmatrix} 1 \\ \cos 2\pi k/n \\ \cos 2\pi 2k/n \\ \vdots \\ \cos 2\pi(n-1)k/n \end{pmatrix},$$

and u_k of the form

$$\begin{pmatrix} 0 \\ \sin 2\pi k/n \\ \sin 2\pi 2k/n \\ \vdots \\ \sin 2\pi(n-1)k/n \end{pmatrix},$$

where k does not run from 1 to n anymore, but from 1 to the first integer less than $n/2$.

Note that all the eigenvalues $2(\cos 2\pi k/n - 1)$ are less than or equal to zero, and only the $k = n = 0$ eigenvalue is equal to 0. The eigenvector associated with eigenvalue 0 is just the vector

$$\begin{pmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix},$$

and thus, conforming to our expectations, the long term probability that the wanderer is on any given node is the same.