Problem Set 8, due Friday, June 5

1. Consider the random walk on an n point circle, except now, the probability that the walker makes a jump clockwise in a time interval $[t, t+h)$ is $2h$, while the probability that the walker makes a jump counterclockwise in a time interval $[t, t+h)$ is h, and the probability of staying put in the time interval $[t, t+h)$ is $1-3h$.

   i: Find a differential equation for the probability that the wanderer is at node j at time t. You do **not** need to prove your equation is correct.

   ii: As $t \mapsto \infty$, what is the probability that the wanderer is at node j? Hint: you will need to look at the real parts of the (complex) eigenvalues of the matrix from your differential equation. Find them by guessing the eigenvectors.

   i: Let $p_j(t)$ be the probability that at time t the wanderer is at node j.

$$\frac{\partial p_j}{\partial t} = \lim_{h \mapsto 0} \frac{p_j(t+h) - p_j(t)}{h}.$$

   By breaking up the history of the walk into the paths longer than 1 jump and the paths 1 jump or less we get the equation

$$p_j(t+h) = hp_{j+1}(t) + 2hp_{j-1}(t) + (1-3h)p_j(t) + o(h),$$

   and so

$$\frac{\partial p_j}{\partial t} = p_{j+1}(t) + 2p_{j-1}(t) - 3p_j(t)$$

   In matrix form, this is the equation $p' = Ap$, with

$$A = \begin{pmatrix} -3 & 1 & 0 & \cdots & 0 & 1 \\ 2 & -3 & 1 & 0 & \cdots & 0 \\ 0 & 2 & -3 & 1 & & \vdots \\ \vdots & & \ddots & & & 0 \\ 0 & & & & & 1 \\ 1 & 0 & \cdots & 0 & 2 & -3 \end{pmatrix}.$$

   ii: A is shift symmetric, so a good guess for the eigevectors is again the vectors $v_0, ..., v_{n-1}$, where the jth component of v_k, $v_k^j = e^{2\pi i(j-1)k/n}$. Are they eigenvectors?

$$(A * v_k)_j = 2e^{2\pi i(j-2)k/n} + e^{2\pi ijk/n} - 3e^{2\pi i(j-1)k/n}$$

$$= 2e^{-2\pi ik/n}e^{2\pi i(j-1)k/n} + e^{2\pi ik/n}e^{2\pi i(j-1)k/n} - 3e^{2\pi i(j-1)k/n}$$
$$= \left(2e^{-2\pi ik/n} + e^{2\pi ik/n} - 3\right) e^{2\pi i(j-1)k/n}$$
$$\left(2e^{-2\pi ik/n} + e^{2\pi ik/n} - 3\right) v_k^j.$$

So they are eigenvectors, with $\lambda_k = 2e^{-2\pi ik/n} + e^{2\pi ik/n} - 3$. The real part of λ_k is less than zero for all $k > 0$, and $\lambda_0 = 0$. The general solution to the equation $p' = Ap$ is

$$\sum_{k=0}^{n-1} C_k e^{\lambda_k t} v_k$$

$$= C_0 v_0 + \sum_{k=1}^{n-1} C_k e^{\lambda_k t} v_k.$$

since v_0 is the vector of all ones, and as t tends to infinity the right hand sum above tends to 0 (all the real parts of λ_k are less than 0 for $k > 0$), the solution tends to $1/n$ times the ones vector, and every node is equally likely.

2. Consider the 5 node network in the figure. Assume that in a time interval $[t, t+h)$, if the wanderer is at the central node, the probability of transfer to each of the outlying nodes is $h/4$, and so the probability of staying put is $1 - h$; if at an outlying node, the probability of a move to the center is h, and so the probability of staying put is $1 - h$.

   i: Find a differential equation for the probability that the wanderer is at node j at time t. You do **not** need to prove your equation is correct.

   ii: As $t \mapsto \infty$, what is the probability that the wanderer is at node j?

i: Using the same method as above, we find the equation to be $p' = Ap$, where

$$A = \begin{pmatrix} -1 & 0 & 0 & 0 & 1/4 \\ 0 & -1 & 0 & 0 & 1/4 \\ 0 & 0 & -1 & 0 & 1/4 \\ 0 & 0 & 0 & -1 & 1/4 \\ 1 & 1 & 1 & 1 & -1 \end{pmatrix}.$$

ii: using a computer, we find the eigenvalues of the above matrix to be 0, -1 with multiplicity 3, and -2. The eigenvector corresponding to 0 is $v_0 = (1, 1, 1, 1, 4)^T$. As usual, as t tends to infinity, all the components of the general solution with negative eigenvalues die, leaving $C_0 v_0$, where $C_0 v_0$ has sum 1. So the long term probabilities are $(1/8, 1/8, 1/8, 1/8, 1/2)^T$.

3. Consider the standard n point random walk $p' = Ap$, where as usual,

$$A = \begin{pmatrix} -2 & 1 & 0 & \cdots & 0 & 1 \\ 1 & -2 & 1 & 0 & \cdots & 0 \\ 0 & 1 & -2 & 1 & & \vdots \\ \vdots & & \ddots & & & 0 \\ 0 & & & & & 1 \\ 1 & 0 & \cdots & 0 & 1 & -2 \end{pmatrix}.$$

   i: Find a nonconstant initial condition x so that if $p(0) = x$, the point on the discrete circle which is the most likely location of the walker does not change in time; i.e. so that the location of maximum value of p is constant in time.

   ii: Find an initial condition z so that if $p(0) = z$, the point on the discrete circle which is the most likely location of the walker does change in time. You may assume the system has a large number of points if it makes it easier for you; note that it is necessary that the number of points be greater than 3.

   For both parts of the question, justify your answers.

Solution: there are many solutions to both parts of this problem; here is one possibility. Denote by x the vector of ones, by w_k the vector with jth position $w_k^j = \cos 2\pi k(j-1)/n$, and by s_k the vector $s_k^j = \sin 2\pi k(j-1)/n$.

for part i, set

$$z = (1/n)x + (1/n)w_1.$$

Then since x and w_1 are eigenvectors for A with eigenvalues 0 and $\lambda_1 = 2\cos 2\pi/n - 2$, if $y(0) = z$,

$$y(t) = (1/n)x + (e^{\lambda_1 t}/n)w_1.$$

But the maximum value for y stays at $j = 1$. For part ii, set

$$z = (1/n)x + (1/100n)w_1 + (1/n)s_2;$$

then

$$y(t) = (1/n)x + (1/100n)e^{\lambda_1 t}w_1 + (1/n)e^{\lambda_2 t}s_2.$$

Then if n is large, at $t = 0$, $y(t)$ has a maximum near $j = \lfloor n/8 \rfloor$; but for large t, since λ_2 is more negative than λ_1,

$$y(t) \sim (1/n)x + (1/100n)e^{\lambda_1 t}w_1,$$

and the maximum is near $j = 1$. For both parts, the trick here is to use the fact that eigenvectors behave simply with respect to the differential equation.