1 A derivation of the heat equation

Consider a thin, insulated wire, of constant density ρ, shaped in a circle of radius 1. We wish to derive an equation for the temperature of the wire $f = f(\theta, t)$ as a function of location and time. A small length of wire $[\theta, \theta + \triangle\theta]$ has mass $\rho\triangle\theta$, where ρ is the density. The wire has specific heat c, which means that a length of wire of mass $\rho\triangle\theta$ and temperature f has heat energy $Q = c\rho f\triangle\theta$. So

$$\frac{\partial Q}{\partial t} = c\rho\frac{\partial f}{\partial t}\triangle\theta.$$

To derive the equation, we suppose the amount of heat energy entering the small length of wire is proportional to the change in temperature at either end of the length of wire, with constant K, which is called the conductivity. In symbols,

$$\frac{\partial Q}{\partial t} = K\left[\frac{\partial f}{\partial \theta}(\theta + \triangle\theta, t) - \frac{\partial f}{\partial \theta}(\theta)\right].$$

The minus sign in the numerator is because we always measure the change in temperature in the direction of increasing θ. We now set the two expressions for $\frac{\partial Q}{\partial t}$ equal to eachother, and obtain:

$$\frac{c\rho}{K}\frac{\partial f}{\partial t} = \frac{\frac{\partial f}{\partial \theta}(\theta + \triangle\theta, t) - \frac{\partial f}{\partial \theta}(\theta)}{\triangle\theta}.$$

Taking limits gives

$$\frac{c\rho}{K}\frac{\partial f}{\partial t}(\theta, t) = \frac{\partial^2 f}{\partial \theta^2}(\theta, t).$$

We now rewrite more cleanly... a subscript denotes a partial derivative in the indicated variable, and for simplicity, we set all constants to 1, obtaining the heat equation:

$$f_t = f_{\theta\theta} \tag{1}$$
$$f(\theta, 0) = f_0. \tag{2}$$

Note that this is an equation which describes the change in the temperature (a function on the circular wire) over time; the solution is a function of time and location. The initial condition is thus a function on the wire.

Notice how this is exactly analagous to the equation we got for the probability distribution of a random walker on an n point circle, which was of the form

$$y_t = Ay \tag{3}$$
$$y(0) = y_0. \tag{4}$$

Here y was a function of two variables (although we did not think of it that way), the discrete space variable and the continuous time variable. The matrix A was just a discrete approximation to the second derivative in the space variable.

2 Some fundamental solutions

We now have the equation (1). How should we solve it? Our finite dimensional random walk approximations to the equation had solutions of the form

$$\sum C_i e^{\lambda_i t} v_i,$$

where C_i where constants that came from the initial condition, v_i were the eigenvectors of the n-dimensional discrete second difference matrix (which were sinusoids), and λ_i were the corresponding eigenvalues. If we think of the vector v_i as a function on a discrete approximation to the circle, the "fundamental" solutions $e^{\lambda_i t} v_i$ all had the form $g(t)h(\theta)$. Can we find solutions of this form in the continuous case? Let's guess we can... Plugging $f = g(t)h(\theta)$ into (1) gives

$$g_t h = g h_{\theta\theta},$$

and rearranging gives

$$\frac{g_t}{g} = \frac{h_{\theta\theta}}{h}.$$

Since the right and left hand sides of this equation depend on different variables, both sides must be constant, and we recover the familiar ODE's

$$g_t = Cg$$

and

$$h_{\theta\theta} = Ch.$$

The equation in h has solutions of the form

$$K_1 e^{\sqrt{C}\theta} + K_2 e^{-\sqrt{C}\theta}$$

if C is positive, and

$$K_1 \cos\sqrt{-C}\theta + K_2 \sin\sqrt{-C}\theta$$

if C is negative. Since we know that as a function of θ, h has to be periodic, of period 2π, we get that $C = -k^2$, for some integer k. The equation for g then has solution $e^{-k^2 t}$, and we get a very familiar expression for f:

$$f(\theta, t) = e^{-k^2 t}\left[K_1 \cos k\theta + K_2 \sin k\theta\right].$$

these are exactly analagous to the eigenvectors of the n point discrete difference operator that we studied before! In fact, it is standard to call sin and cos eigenfunction of the second derivative.

3 Fourier series

Given an initial function f_0, we would like to use the fundamental solutions constructed in the previous sections to build a solution to (1) and (2). In the finite dimensional case, the solution was just to form

$$\sum C_i e^{\lambda_i t} v_i,$$

where the C_i were the coefficients of v_0 in the basis $v_1, \cdots, v_n$. The same approach will work now, but the sums are all infinite. That is: suppose

$$f_0 = a_0 + \sum_{k=1}^{\infty} a_k \cos k\theta + \sum_{k=1}^{\infty} b_k \sin k\theta. \tag{5}$$

Then the solution to (1) and (2) is simply

$$f(\theta, t) = a_0 + \sum_{k=1}^{\infty} e^{-k^2 t} \left[a_k \cos k\theta + b_k \sin k\theta \right]. \tag{6}$$

This expansion of f_0 in (5) is called the Fourier series for f_0. When can we expand a function in a Fourier series? When does the series converge? What does it mean for the series to converge? How can we find the coefficients a_k and b_k? The answer to these questions are slightly complicated, and we will not be able to completely answer them. However, we now try to give some non-rigorous elements of a partial answer.

We first answer the easier question of how to find the coefficients in the expansion, and start by a quick review of some linear algebra. Recall that the inner product $\langle x, y \rangle$ between two n-vectors x and y is defined by

$$\langle x, y \rangle = \sum_{k=1}^{n} x_k y_k. \tag{7}$$

The inner product measures the angle between two vectors; so if $\langle x, y \rangle = 0$, the vectors are orthogonal. If $\{v_k\}$ is a basis for $\mathbb{R}^n$; and $\langle v_k, v_j \rangle = 0$ if $k \neq j$, and $\langle v_k, v_k \rangle = ||v_k||^2 = 1$, $\{v_k\}$ is said to be orthonormal. The coordinates of a vector x in $\{v_k\}$ are then easy to find:

$$x = \sum_{k=1}^{n} \langle x, v_k \rangle v_k$$

As n tends to ∞, the sum in (7) tends to an integral; and if x and y are functions on the circle, define

$$\langle x, y\rangle = \int_0^{2\pi} x(\theta)y(\theta)\ d\theta. \tag{8}$$

It can be checked using the angle addition formulas that

$$\int_0^{2\pi} \cos k\theta \cos j\theta\ d\theta = 0 \text{ if } k \neq j \tag{9}$$
$$\int_0^{2\pi} \sin k\theta \sin j\theta\ d\theta = 0 \text{ if } k \neq j \tag{10}$$
$$\int_0^{2\pi} \cos k\theta \sin j\theta\ d\theta = 0 \tag{11}$$
$$\int_0^{2\pi} \cos^2 k\theta\ d\theta = \pi \tag{12}$$
$$\int_0^{2\pi} \sin^2 j\theta\ d\theta = \pi \tag{13}$$

These formulas say that the cosine and sine functions are orthogonal. Thus if (5) holds, and the sum converges absolutely,

$$\langle f_0, \cos j\theta\rangle = \int_0^{2\pi} f_0(\theta)\cos j\theta\ d\theta$$

$$= a_0 \int_0^{2\pi} \cos j\theta\ d\theta + \sum_{k=1}^{\infty} a_k \int_0^{2\pi} \cos j\theta \cos k\theta\ d\theta + \sum_{k=1}^{\infty} b_k \int_0^{2\pi} \cos j\theta \sin k\theta\ d\theta$$

$$= \pi a_j,$$

where we used the fact that the series converges absolutely to interchange the integrals and sums. The third line follows from the second because all the integral terms in the second line are 0 except $\int_0^{2\pi} \cos j\theta \cos j\theta\ d\theta$. This means that to find a_j of a function f_0 whose Fourier series converges absolutely, it suffices to take the inner product between f_0 and $\cos j\theta$. The same holds for b_j, which is obtained by taking inner products with $\sin j\theta$.

When does (5) converge absolutely? This is a slightly more delicate question. However, we have the following sufficient condition: if f_0 has a continuous derivative, the sum converges absolutely. We will not prove this here (although it is not too hard...). We also give the following as an exercise (not to turn in; only if you like this sort of thing!): show that if f_0 has two continuous derivatives, (5) converges absolutely. Hint: using integration by parts twice, we have the formula $\langle f_0, \cos j\theta\rangle = -\langle f_0'', \cos j\theta\rangle / j^2$

There is one more subtlety we have glossed over. Is it possible that the sum in (5) converges absolutely, but f_0 is not equal to the sum? By the linearity of the inner product, this amounts to finding a function $g \neq 0$, but

$\langle g, \cos j\theta \rangle = 0$ and $\langle g, \sin j\theta \rangle = 0$ for all j. The answer is that there is no such (smooth) function g; again, this is not too hard, but we will not show it here. Thus if f_0 is a function with a continuous derivative, the equality (5) holds, and (6) is the solution to (1) with initial condition (2).