Then

$$a_1^T p = -4 < 0, \quad a_2^T p = -6 < 0, \quad \text{and} \quad a_3^T p = 18 > 0,$$

so only the first two constraints are used in the ratio test:

$$\begin{aligned} \bar{\alpha} &= \min \left\{ (a_i^T \bar{x} - b_i)/(-a_i^T p) : a_i^T p < 0 \right\} \\ &= \min \{ (5-3)/4, (3-2)/6 \} = 1/6. \end{aligned}$$

Notice that the point $\bar{x} + \bar{\alpha} p = (\frac{5}{3}, \frac{2}{3})^T$ is on the boundary of the second constraint.

## EXERCISES

1. Find the sets of all feasible directions at points $x_a = (0, 0, 2)^T$, $x_b = (3, 0, 1)^T$, and $x_c = (1, 1, 1)^T$ for Example 3.1.

2. Consider the set defined by the constraints $x_1 + x_2 = 1$, $x_1 \geq 0$ and $x_2 \geq 0$. At each of the following points determine the set of feasible directions: (a) $(0, 1)^T$; (b) $(1, 0)^T$; (c) $(0.5, 0.5)^T$.

3. Consider the system of inequality constraints $Ax \geq b$ with

$$A = \begin{pmatrix} 9 & 4 & 1 & 9 & -7 \\ 6 & -7 & 8 & -4 & -6 \\ 1 & 6 & 3 & -7 & 6 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} -15 \\ -30 \\ -20 \end{pmatrix}.$$

For the given values of $x$ and $p$, perform a ratio test to determine the maximum step length $\bar{\alpha}$ such that $x + \bar{\alpha} p$ remains feasible.
(*a*) $x = (8, 4, -3, 4, 1)^T$ and $p = (1, 1, 1, 1, 1)^T$
(*b*) $x = (7, -4, -3, -3, 3)^T$ and $p = (3, 2, 0, 1, -2)^T$
(*c*) $x = (5, 0, -6, -8, -3)^T$ and $p = (5, 0, 5, 1, 3)^T$
(*d*) $x = (9, 1, -1, 6, 3)^T$ and $p = (-4, -2, 4, -2, 2)^T$

4. What are the potential consequences of miscalculating $\bar{\alpha}$ in the ratio test?

5. Let $S = \{x : Ax \leq b\}$. Derive the conditions that must be satisfied by a feasible direction at a point $\bar{x} \in S$.

6. On a computer, there is a danger that an overflow can occur during the ratio test if, in a particular ratio, the numerator is large and the denominator is small. How can the ratio test be implemented so that this danger is removed?

## 3.2 NULL AND RANGE SPACES

Let $A$ be an $m \times n$ matrix with $m \leq n$. We denote the *null space* of $A$ by

$$\mathcal{N}(A) = \left\{ p \in \Re^n : Ap = 0 \right\}.$$

## EXERCISES

1. Solve the following linear programs graphically

(*a*)

$$\begin{array}{ll} \text{minimize} & z = 3x_1 + x_2 \\ \text{subject to} & x_1 - x_2 \le 1 \\ & 3x_1 + 2x_2 \le 12 \\ & 2x_1 + 3x_2 \le 3 \\ & -2x_1 + 3x_2 \ge 9 \\ & x_1, x_2 \ge 0. \end{array}$$

(*b*)

$$\begin{array}{ll} \text{maximize} & z = x_1 + 2x_2 \\ \text{subject to} & 2x_1 + x_2 \ge 12 \\ & x_1 + x_2 \ge 5 \\ & -x_1 + 3x_2 \le 3 \\ & 6x_1 - x_2 \ge 12 \\ & x_1, x_2 \ge 0. \end{array}$$

(*c*)

$$\begin{array}{ll} \text{minimize} & z = x_1 - 2x_2 \\ \text{subject to} & x_1 - 2x_2 \ge 4 \\ & x_1 + x_2 \le 8 \\ & x_1, x_2 \ge 0. \end{array}$$

(*d*)

$$\begin{array}{ll} \text{minimize} & z = -x_1 - x_2 \\ \text{subject to} & x_1 - x_2 \ge 1 \\ & x_1 - 2x_2 \ge 2 \\ & x_1, x_2 \ge 0. \end{array}$$

(*e*)

$$\begin{array}{ll} \text{minimize} & z = x_1 - x_2 \\ \text{subject to} & x_1 - x_2 \ge 2 \\ & 2x_1 + x_2 \ge 1 \\ & x_1, x_2 \ge 0. \end{array}$$

(*f*)

$$\begin{array}{ll} \text{minimize} & z = 4x_1 - x_2 \\ \text{subject to} & x_1 + x_2 \le 6 \\ & x_1 - x_2 \ge 3 \\ & -x_1 + 2x_2 \ge 2 \\ & x_1, x_2 \ge 0. \end{array}$$

(*g*)

$$\begin{array}{ll} \text{maximize} & z = 6x_1 - 3x_2 \\ \text{subject to} & 2x_1 + 5x_2 \ge 10 \\ & 3x_1 + 2x_2 \le 40 \\ & x_1, x_2 \le 15. \end{array}$$

## EXERCISES

1. Convert the following linear program to standard form

$$\begin{array}{ll} \text{maximize} & z = 3x_1 + 5x_2 - 4x_3 \\ \text{subject to} & 7x_1 - 2x_2 - 3x_3 \geq 4 \\ & -2x_1 + 4x_2 + 8x_3 = -3 \\ & 5x_1 - 3x_2 - 2x_3 \leq 9 \\ & x_1 \geq 1,\ x_2 \leq 7,\ x_3 \geq 0. \end{array}$$

2. Convert the following linear program to standard form

$$\begin{array}{ll} \text{minimize} & z = x_1 - 5x_2 - 7x_3 \\ \text{subject to} & 5x_1 - 2x_2 + 6x_3 \geq 5 \\ & 3x_1 + 4x_2 - 9x_3 = 3 \\ & 7x_1 + 3x_2 + 5x_3 \leq 9 \\ & x_1 \geq -2,\ x_2, x_3 \text{ free.} \end{array}$$

3. Convert the following linear program to standard form

$$\begin{array}{ll} \text{maximize} & z = 6x_1 - 3x_2 \\ \text{subject to} & 2x_1 + 5x_2 \geq 10 \\ & 3x_1 + 2x_2 \leq 40 \\ & x_1, x_2 \leq 15. \end{array}$$

4. Consider the linear program in Example 4.2. Convert it to standard form, except do not make the substitution $x_3 = x_3' - x_3''$. Show that the problem can be replaced by an equivalent problem with one less variable and one less constraint by eliminating $x_3$ using the equality constraints. (This is a general technique for handling free variables.) Why cannot this technique be used to eliminate variables with nonnegativity constraints?

5. Consider the linear program

$$\begin{array}{ll} \text{minimize} & z = c^T x \\ \text{subject to} & Ax \leq b \\ & e^T x = 1 \\ & x_1, \ldots, x_{n-1} \geq 0,\ x_n \text{ free} \end{array}$$

where $e = (1, \ldots, 1)^T$, $b$ and $c$ are arbitrary vectors of length $n$, and $A$ is the matrix with entries $a_{i,i} = a_{i,n} = 1$ for $i = 1, \ldots, n$ and all other entries zero. Use the constraint $e^T x = 1$ to eliminate the free variable $x_n$ from the linear program (as in the previous problem). Is this a good approach when $n$ is large?

6. Prove that each of the transformation rules used to convert a linear program to standard form produces an equivalent linear programming problem. Hint: For each of the rules, prove that a solution to the original problem can be used to obtain a solution to the transformed problem, and vice versa.

7. Consider the linear program

$$\begin{array}{ll} \text{minimize} & z = c^T x \\ \text{subject to} & Ax = b \\ & x \geq 0. \end{array}$$

it is possible to select two linearly independent directions of unboundedness, such as $d_1 = (1,0)^T$ and $d_2 = (1,1)^T$. It is not difficult to show that any feasible point can be written as a convex combination of the extreme points $x_a$, $x_b$, and $x_c$, plus some multiple of either of these directions of unboundedness.

Let $x$ be a feasible point for the linear program in standard form ($Ax = b$, $x \geq 0$) and let $d$ be a direction of unboundedness. Then both $x$ and $x + \gamma d$ must be feasible for all $\gamma \geq 0$, so that

$$Ax = b, \quad x \geq 0,$$
$$A(x + \gamma d) = b, \quad x + \gamma d \geq 0.$$

Together these conditions show that a direction of unboundedness must satisfy

$$Ad = 0$$
$$d \geq 0.$$

In addition, any nonzero vector $d$ satisfying these two conditions will be a direction of unboundedness; see the Problems.

## EXERCISES

1. Consider the system of linear constraints

$$\begin{aligned} 2x_1 + x_2 &\leq 100 \\ x_1 + x_2 &\leq 80 \\ x_1 \qquad &\leq 40 \\ x_1, x_2 &\geq 0. \end{aligned}$$

(*a*) Write this system of constraints in standard form, and determine all the basic solutions (feasible and infeasible).

(*b*) Determine the extreme points of the feasible region (corresponding to both the standard form of the constraints, as well as the original version).

2. Consider the following system of inequalities:

$$\begin{aligned} x_1 + x_2 &\leq 5 \\ x_1 + 2x_2 &\leq 6 \\ x_1, x_2 &\geq 0. \end{aligned}$$

(*a*) Find the extreme points of the region defined by these inequalities.

(*b*) Does this set have any directions of unboundedness? Either prove that none exists, or give an example of a direction of unboundedness.

3. Consider the feasible region in Figure 4.5.

(*a*) Show that $d_1 = (1,0)^T$ and $d_2 = (1,1)^T$ are directions of unboundedness. Determine the corresponding directions of unboundedness for the problem written in standard form, and verify that the conditions $Ad = 0$ and $d \geq 0$ are satisfied for both directions.

(*b*) Prove that $d$ is a direction of unboundedness if and only if $d$ is a nonnegative combination of $d_1$ and $d_2$.

4. Consider the linear program

$$\begin{aligned} \text{minimize} \quad & z = -5x_1 - 7x_2 \\ \text{subject to} \quad & -3x_1 + 2x_2 \leq 30 \\ & -2x_1 + x_2 \leq 12 \\ & x_1, x_2 \geq 0. \end{aligned}$$

(*a*) Draw a graph of the feasible region.
(*b*) Determine the extreme points of the feasible region.
(*c*) Determine two linearly independent directions of unboundedness.
(*d*) Convert the linear program to standard form and determine the basic feasible solutions and two linearly-independent directions of unboundedness for this version of the problem. Verify that the directions of unboundedness satisfy $Ad = 0$ and $d \geq 0$.

5. Consider a linear program with the constraints in standard form

$$Ax = b \quad \text{and} \quad x \geq 0.$$

Prove that if $d \neq 0$ satisfies

$$Ad = 0 \quad \text{and} \quad d \geq 0$$

then $d$ is a direction of unboundedness.

6. Consider the system of constraints

$$\begin{aligned} 2x_1 + x_2 &\leq 3 \\ 3x_1 + x_2 &\leq 4 \\ 4x_1 + x_2 &\leq 5 \\ 5x_1 + x_2 &\leq 6 \\ x_1, x_2 &\geq 0. \end{aligned}$$

(*a*) Determine the extreme points for the feasible region.
(*b*) Convert the problem to standard form, and determine the basic feasible solutions.
(*c*) Which basic feasible solution corresponds to the extreme point $(1,1)^T$? How many different bases can be used to generate this basic feasible solution? Which of these bases are adjacent?

7. Consider the feasible region in Figure 4.4. Determine formulas for the points on the edges of the feasible region. What are the corresponding formulas for the problem in standard form? The formulas you determine should be of the form

$$(\text{extreme point}) + \alpha(\text{direction}), \quad \text{for } 0 \leq \alpha \leq \alpha_{max}.$$

8. Repeat the previous problem for the feasible region in Figure 4.5. Note that in some cases there will be no upper bound on $\alpha$.

9. Consider the system of constraints $Ax = b$, $x \ge 0$ with

$$A = \begin{pmatrix} 1 & 4 & 7 & 1 & 0 & 0 \\ 2 & 5 & 8 & 0 & 1 & 0 \\ 3 & 6 & 9 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} 12 \\ 15 \\ 18 \end{pmatrix}.$$

Is $x = (1, 1, 1, 0, 0, 0)^T$ a basic feasible solution? Explain your answer.

10. Suppose that a linear program includes a free variable $x_i$. In converting this problem to standard form, $x_i$ is replaced by a pair of non-negative variables:

$$x_i = x_i' - x_i'', \quad x_i', x_i'' \ge 0.$$

Prove that no basic feasible solution can include both $x_i'$ and $x_i''$ as basic variables.

11. Let the $m \times n$ matrix $A$ be the coefficient matrix for a linear program in standard form. The upper bound

$$\binom{n}{m} = \frac{n!}{m!(n-m)!}$$

on the number of basic feasible solutions can sometimes be precise, but it can also be a considerable overestimate.

(*a*) Construct an example with $n = 4$ and $m = 2$ where the number of basic feasible solutions is equal to $\binom{n}{m}$.

(*b*) Construct examples of arbitrary size where the number of basic feasible solutions is equal to zero.

12. Prove that the set $S = \{x : Ax < b\}$ does not contain any extreme points.

13. Let $S = \left\{x : x^T x \le 1\right\}$. Prove that the extreme points of $S$ are the points on its boundary.

14. Consider the set $S = \left\{x : x_1 \ge x_2 \ge \cdots \ge x_n \ge 0\right\}$.

(*a*) Prove that if $x \in S$ then so is $\alpha x \in S$ for all $\alpha \ge 0$. A set with this property is called a *cone*.

(*b*) Prove that the origin is the only extreme point of $S$.

(*c*) Find $n$ linearly independent directions of unboundedness for this set.

## 4.4 REPRESENTATION OF SOLUTIONS; OPTIMALITY

The first goal of this section is to prove that any feasible point can be represented as a convex combination of extreme points plus, possibly, a direction of unboundedness. Then this result will be used to prove that any linear program with a finite optimal solution has an optimal basic feasible solution.

The idea behind the representation theorem is straightforward and will first be illustrated using two examples of feasible sets, one bounded and one unbounded. The examples will be in two dimensions so that they can be graphed, but the techniques used in the examples are the same as those used in the proof.