Hebrew University of Jerusalem
האוניברסיטה העברית בירושלים

Ultrapowers of Forcing Notions
על־חזקות של מושגי כפייה

by

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הקדמה

שיטת הכפייה הומצאה (או לחלופין, התגלתה) ע"י פול כהן בשנות ה60. השיטה פותחה על מנת להוכיח את העקביות של אקסיומות תורת הקבוצות, ZFC, ביחד עם שלילת השערת הרצף, ובכך ליישב את הבעיה הראשונה ברשימת 23 הבעיות של הילברט: השערת הרצף הינה בלתי תלויה באקסיומות של תורת הקבוצות. לאחר הוכחתו של כהן, שיטת הכפייה הפכה במהרה לכלי חשוב ומרכזי בתורת הקבוצות. השיטה סללה דרך להוכחות אי־תלות רבות, ובכך לפתרון של בעיות פתוחות רבות. בנוסף על כך, נמצאו שימושים אחרים לשיטת הכפיית, לא רק עבור תוצאות אי־תלות. כתוצאה מכך, גבר העניין הפנימי בשיטת הכפייה עצמה, וכן בתכונות וטכניקות של כפייה שעלו מן ההוכחות.

בעבודה זו נתעניין בהתנהגות של מושגי כפייה תחת פעולת העל־חזקה. המקרה בו נתמקד הוא כדלקהלן: יהי λ מונה מדיד ו־$\mathscr{U}$ על־מסנן נורמלי מעל λ. נסמן ב $j:V \longrightarrow N$ את שיכון העל־חזקה הנגזר מ־$\mathscr{U}$, כאשר N הוא הטלת מוסטובסקי של מודל העל־חזקה $\mathrm{Ult}(V,\mathscr{U})$. בהנתן מושג כפייה P, נתעניין בכפיית העל מכפלה $j(P)$. ראשית נשים לב כי, על פי האלמנטריות של j, הכפייה $j(P)$ מקיימת את אותן התכונות בN כפי ש־P מקיימת ב־V. עם זאת, אנו לא נתעניין ב$j(P)$ כמושג כפייה בN, אלא כמושג כפייה בV. אכן נראה כי התכונות של $j(P)$ ככפייה מעל V יכולות להיות שונות מאוד מאלו של P. נשים לב כי הכפייה $j(P)$, כסדר חלקי, הינה פשוט העל מכפלה של P ע"י $\mathscr{U}$ במובן התורת־מודלי, דהיינו $P^{\lambda}/\mathscr{U}$. מאידך, הכפיות P בהן נעסוק הן גדירות באופן פשוט יחסית, לכן ניתן לחשוב על $j(P)$ כעל הסיפוק של אותה ההגדרה עפ"י המודל N.

המקרה המרכזי בו נתעניין הוא כאשר הכפייה P "כופה מעל λ". התכונה המדוייקת בה נשתמש הינה שהכפייה P^{λ} היא λ^{+}־דיסטריביוטיבית. חשיבותה העקרית של דרישה זו נעוצה בכך שגם לאחר הכפיות הרלוונטיות, $\mathscr{U}$ נותר על־מסנן מעל λ, וכן שיכון העל־חזקה החדש מרחיב את השיכון המקורי (זאת, כיוון שלא נוספו סדרות חדשות באורך λ).

נתעניין באילו תכונות עוברות מהכפייה P ל־$j(P)$. נעסוק בתכונות כגון סגירות, סגירות אסטרטגית ודיסטריביוטיביות. בנוסף, נחקור את העל־חזקה של מספר כפיות מוכרות היטב, ביניהן הכפייה ההורסת קבוצת שבת וכפייה התופרת סדרת ריבוע. במיוחד נתעניין בעל־חזקה של כפיית קבוצת כהן במונה סדיר κ, ובהשפעתה על מונים גדולים.

להלן פירוט קצר של תוכן העבודה:

תחילה, בסעיף ההקדמה נסקור כמה מושגים בסיסיים בכפייה ובעל מכפלות. נרחיב בפירוט על שני הנושאים הבאים, שנעשה בהם שימוש חוזר ונשנה לאורך העבודה: הטלות בין מושגי כפייה והרחבות של שיכוני על־חזקה לאחר כפייה.

בסעיף 1, נדון באופן כללי בעל־חזקה של כפייה P, $j(P)$. נראה כי ניתן לשכן את כפיית העל־חזקה בכפיית החזקה, P^{λ}. מכך נסיק בפרט שדיסטריביוטיביות של P^{λ} גוררת דיסטריביוטיביות של $j(P)$, וכן שסגירות אסטרטגית עוברת מ־P ל־$j(P)$. כמוכן נראה כי אם H גנרי עבור $j(P)$ מעל V, אזי H ו־$j(H)$ גנרים הדדית מעל N.

בסעיף 2, ניתן דוגמא לכפייה P כך ש־P היא κ־דיסטריביוטיבית אך $j(P)$ אינה. מכך נובע כי הדרישה על P^{λ} בסעיף 1 היא מהותית. הכפיה P בדוגמא שניתן היא הכפייה התופרת סדרת ריבוע. הסעיף כולל הגדרות של כל המושגים הרלוונטים וכן הוכחות של עובדות בסיסיות הנחוצות לנו.

בסעיף 3, נקח את P להיות כפיית קבוצת כהן במונה סדיר $\kappa < \lambda$. נתעניין בעל־חזקה של P, במיוחד בהקשר של שימור והריסת מונים גדולים. נראה כי $j(P)$ הורסת את הקומפקטיות החלשה של κ, וכן הורסת קומפקטיות חזקה מתחת ל־κ. ליתר דיוק, נגדיר אובייקט קומבינטורי מסויים הסותר קומפקטיות, ונראה כי הכפייה $j(P)$ מוסיפה אובייקט זה. נחקור גם את כפיית המנה $P^{\lambda}/j(P)$.

בסעיף 4, נעסוק בשאלה הבאה: "האם כפייה עם $j(P)$ מוסיפה גנרי עבור P, מעל עולם הבסיס?". ניתן דוגמאות לכאן ולכאן. בפרט נדון בעל־חזקה של הכפייה ההורסת קבוצת שבת.

CONTENTS

INTRODUCTION

Let $\mathscr{U}$ be a normal ultrafilter over a measurable cardinal λ, and consider the corresponding elementary embedding $j\colon V \longrightarrow N$, where N is the transitive collapse of $\mathrm{Ult}(V,\mathscr{U})$. For a forcing notion $Q \in N$, we want to add a Q-generic, by forcing with Q as a forcing notion in V. In the generic extension, the Q-generic over V is in particular Q-generic over N. Note that N is elementarily equivalent to V, that is, N is in some sense a "close approximation" of the entire universe V. However, the properties of Q as a forcing notion in V can be quite different from its properties as a forcing notion in N.

We focus on the case $Q = j(P)$, where P is a forcing notion in V. P will typically be a forcing notion which is familiar and well understood, hence by elementarity, $j(P)$ is also well understood, as a forcing notion in N. We then study its properties as a forcing notion in V. Note that in this case, $j(P)$ is equal to $P^{\lambda}/\mathscr{U}$, the ultrapower of P by $\mathscr{U}$. Furthermore, the forcing notion P usually satisfies some simple definition in V, hence $j(P)$ is simply the satisfaction of this definition in the model N.

We will be primarily interested in forcing notions P that "force above λ". The accurate assumption is that P^{λ} is λ^{+}-distributive; the practical conclusion is that in the forcing extension, $\mathscr{U}$ remains an ultrafilter, and the new ultrapower embedding extends the old one (as no new λ-sequences of elements from V are added). This assumption will prove useful in the study of $j(P)$-generic extensions.

We study the ultrapower of some familiar forcing notions, such as the forcing notion for destroying a stationary subset, and the forcing notion for threading a square sequence. Particular attention will be given to the ultrapower of Cohen forcing, and its interaction with large cardinals.

We remark that ultrapowers of forcing notions were used by Shelah in the context of cardinal invariants of the continuum (see [1]). However, in Shelah's proof, the ultrapower of a c.c.c forcing notion that blows up the continuum to λ was used. Thus the situation described above is very different in nature.

A short description of the main contents of each section is as follows:

In the preliminaries section, we mostly review basic facts and definitions about forcing and ultrapower embeddings. A more detailed discussion about projections of forcing notions and ultrapower embeddings of generic extensions is included, as these will be substantially used throughout this paper.

In section 1, we study $j(P)$ for a general poset P. We show that as forcing notions, $j(P)$ can be embedded in P^{λ}, and conclude that distributivity is transferred from P^{λ} to $j(P)$. We also show that if $H \subset j(P)$ is generic over V, then H and $j(H)$ are mutually generic over N.

In section 2, we give an example for a poset P which is κ-distributive, yet $j(P)$ is not κ-distributive. Hence the assumption on P^{λ} in section 1 is essential. The example is the forcing notion for threading a $\square$-sequence.

In section 3, we take P to be Cohen forcing at some regular cardinal $\kappa > \lambda$. We then use the results of section 1 to study $j(P)$ in the context of destruction and preservation of large cardinals. We show that $j(P)$ in fact destroys the weak compactness of κ. More explicitly, we define a certain combinatorial principle which contradicts compactness, and show that $j(P)$ forces this principle. We also study the quotient forcing $P^{\lambda}/j(P)$.

In section 4, we consider the following question: "is a P-generic added when forcing with $j(P)$?". We give examples for either direction. The forcing notion for destroying a stationary subset is considered here.

PRELIMINARIES

We assume the reader is familiar with forcing and ultrapower embeddings. This preliminary section does not pretend to introduce these concepts, but rather will review some basic definitions and facts, and fix some notation. We focus on stating facts and properties that will be used later rather than giving accurate definitions. We refer to any of the common textbooks on set theory and forcing, for example [4, 6, 5].

0.1. **Forcing.** We follow the usual approach for forcing with partial orders. A triplet $(P, \leq_P, 1)$ is said to be a forcing notion, or a poset, if $(P, \leq_P)$ is a partially ordered set with a maximal element $1 \in P$. We follow the convention that for $p, q \in P$, p *extends* q (or p is *stronger* than q) if $p \leq_P q$.
The forcing notions we work with are assumed to be separative, so if $p \Vdash q \in \dot{G}$, where $\dot{G}$ is the name for the generic filter, then $p \leq q$. In some cases we may deal with posets that are not separative by definition, but we always have the separative quotient in mind, which produces equivalent forcing extensions as the original poset.
One of the most basic forcing notions is Cohen forcing for a cardinal κ. The κ-Cohen poset is defined as follows:

$$\text{Cohen}(\kappa) = P_\kappa = \{p : \text{dom} p \longrightarrow 2; |\text{dom} p| < \kappa\},$$

where for $p, q \in P_\kappa$, $p \leq q \iff p \supseteq q$.
We sometimes also associate Cohen forcing at κ with the poset $\{p : \text{dom} p \longrightarrow 2; \text{dom} p \in \kappa\}$, which is equivalent as a forcing notion to P_κ, since it is dense in P_κ.
For Cohen forcing, as well as other forcing notions which are composed of partial functions into 2, a condition $p \in P_\kappa$ is associated with the bounded set $\{\alpha \in \text{dom} p; p(\alpha) = 1\}$. Similarly, a generic filter $G \subset P_\kappa$ is associated with the *generic function* $\bigcup G \colon \kappa \longrightarrow 2$ as well as the *generic subset* of κ, $\{\alpha < \kappa; \bigcup G(\alpha) = 1\}$. When writing G we may refer to either the function or the subset, depending on the context.

0.1.1. *Closure and distributivity properties.*

Definition. P is κ-closed if for any descending chain of conditions of length $<\kappa$ there is a lower bound. That is, if $\langle p_\alpha ; \alpha<\eta\rangle \subset P$, $\alpha<\beta<\eta \implies p_\beta \leq p_\alpha$ and $\eta<\kappa$, then there is $p\in P$ s.t $\forall\alpha<\eta\,(p\leq p_\alpha)$.

Definition. P is κ-directed closed if any directed set of conditions of size $<\kappa$ has a lower bound. That is, if $F\subset P$, $|F|<\kappa$ and $\forall p,q\in F\exists r\in F\,(r\leq p\wedge r\leq q)$, then there is $p\in P$ s.t $\forall q\in F\,(p\leq q)$.

Definition. P is κ-distributive if forcing with P adds no new subsets of ordinals of size $<\kappa$.

Fact 0.1. *Let P be a separative poset. The following are equivalent:*

(1) P is κ-distributive.
(2) The intersection of $<\kappa$ many dense subsets of P is dense.

Next we consider strategic closure. We follow the definitions in [2].

Definition. For an ordinal α and poset P, define $G_\alpha(P)$ to be the following two players game of length α.
Player Even plays at even stages (including limit stages) and player Odd plays at odd stages. $p_0 = 1_P$, and at each stage β, the player picks a $p_\beta\in P$ such that $\forall\gamma<\beta\,(p_\beta\leq p_\gamma)$, otherwise the player loses. Player Even wins the play iff Player Even does not lose at any stage $\beta<\alpha$.

Definition 0.2. P is κ-strategically closed if player Even has a winning strategy for the game $G_\kappa(P)$.

The following implications hold and are known to be strict implications:

$$\kappa\text{-closure} \implies \kappa\text{-strategic closure} \implies \kappa\text{-distributivity}.$$

We will see examples for forcing notions separating these properties in the following sections.

0.1.2. *Product forcing and two step iterations.*

Definition. ([6, p.252]) Let $(P,\leq_P,1_P),(Q,\leq_Q,1_Q)\in V$ be forcing notions. We define the product forcing notion $(P\times Q,\leq,1)$ as follows. $1=(1_P,1_Q)$ and for $(p_1,q_1),(p_2,q_2)\in P\times Q$,

$$(p_1,q_1)\leq(p_2,q_2) \iff p_1\leq p_2\wedge q_1\leq q_2.$$

The most important fact about product forcing is that it is the same as iterating the forcings one after another, and that the factors commute, which is captured by the following.

Fact 0.3. *(See* [6, Ch. VIII]*) Suppose $P,Q\in V$, $G\subset P$, $H\subset Q$; then the following are equivalent:*

- $G\times H$ is $P\times Q$-generic over V.
- G is P-generic over V and H is Q-generic over $V[G]$.
- H is Q-generic over V and G is P-generic over $V[H]$.

Furthermore, if the above conditions hold then

$$V[G\times H]=V[G][H]=V[H][G].$$

In this case we say that G and H are mutually generic over V.

Similarly, we can take products of more than two posets. Suppose λ is an ordinal and for each $\alpha<\lambda$ we have a poset P_α. Define

$$P=\prod_{\alpha<\lambda}P_\alpha=\{f;\operatorname{dom}f=\lambda\wedge\forall\alpha<\lambda\,(f(\alpha)\in P_\alpha)\}.$$

If $G\subset P$ is P-generic, then for each $\alpha<\lambda$, $G_\alpha=\{f(\alpha);f\in G\}$ is P_α-generic. Furthermore, by fact 0.3 we get that for each $\alpha\neq\beta$, G_α and G_β are mutually generic. Note also that $f\in G\iff\forall\alpha<\lambda\,(f(\alpha)\in G_\alpha)$, that is, $G=\prod_{\alpha<\lambda}G_\alpha$.

Definition. ([4, p.267]) Let $(P,\leq_P,1_P)\in V$ be a forcing notion and $\left(\dot{Q},\dot{\leq}_Q,\dot{1}_Q\right)$ a P-name for a forcing notion. Define a forcing notion $\left(P*\dot{Q},\leq,1\right)$ as follows:

- $P*\dot{Q}=\left\{(p,\dot{q});p\in P\wedge\Vdash_P\dot{q}\in\dot{Q}\right\}$, $1=\left(1_P,\dot{1}_Q\right)$.
- $(p_1,\dot{q}_1)\leq(p_2,\dot{q}_2)\iff p_1\leq p_2\wedge p_1\Vdash\dot{q}_1\leq\dot{q}_2$.

Fact. *(*[4, p.267]*) Let P be a forcing notion and $\dot{Q}$ a P-name for a forcing notion.*

(1) If G is P-generic over V and H is $\dot{Q}^G$-generic over $V[G]$, then $G*H$ is $P*\dot{Q}$-generic over V, where

$$G*H=\left\{(p,\dot{q});p\in G\wedge\dot{q}^G\in H\right\}.$$

(2) If K is $P*\dot{Q}$-generic over V, define

$$G=\{p\in P;\exists\dot{q}\,(p,\dot{q})\in K\},\quad H=\left\{\dot{q}^G;\exists p\,(p,\dot{q})\in K\right\}.$$

Then G is P-generic over V, H is $\dot{Q}^G$-generic over $V[G]$, and $K=G*H$.

Furthermore, in the situation above we have $V[G*H]=V[G][H]$.

0.1.3. *Embeddings and projections.*

Definition 0.4. ([6, p. 218]) Let P,Q be posets. A map $i\colon P\longrightarrow Q$ is called a *complete embedding* if

(1) $\forall p_1,p_2\in P\,(p_1\leq p_2\implies i(p_1)\leq i(p_2))$.
(2) $\forall p_1,p_2\in P\,(p_1\perp p_2\implies i(p_1)\perp i(p_2))$.
(3) $\forall q\in Q\exists p\in P\forall p'\in P\,(p'\leq p\implies (i(p')$ and q are compatible in $Q))$.

Given 1. and 2., condition 3. is equivalent to the assertion that for any maximal antichain $A\subset P$, $i''A$ is a maximal antichain in Q.
The important fact is that if $i\colon P\longrightarrow Q$ is a complete embedding, then in a Q-generic extension there is also a P-generic filter. More accurately, Q can be thought of as a two step iteration, where the first factor is P. That is, there is a P-name for a forcing notion $\dot{R}$ such that $P*\dot{R}$ is forcing isomorphic to Q.

Definition 0.5. ([6, p. 221]) Let P, Q be posets. A map $i\colon P\longrightarrow Q$ is said to be a *dense embedding* if

(1) $\forall p,p'\in P\,(p'\leq p\implies i(p')\leq i(p))$.
(2) $\forall p,p'\in P\,(p\perp p'\implies i(p)\perp i(p'))$.
(3) $i''P$ is dense in Q.

A dense embedding is in particular a complete embedding. The important fact is that if $i\colon P \longrightarrow Q$ is a dense embedding, then P and Q produce the same generic extensions. In this case we say that P and Q are forcing isomorphic or forcing equivalent.

Dually to complete embeddings, we have complete projections.

Definition 0.6. Let P, Q be posets. A map $a\colon P \longrightarrow Q$ is said to be a *complete projection* if

- $\forall p_1, p_2 \in P\,(p_1 \leq p_2 \implies a(p_1) \leq a(p_2))$ (a is a homomorphism).
- $\forall p \in P \forall q \in Q\,(q \leq a(p) \implies \exists p' \in P\,(p' \leq p \wedge a(p') \leq q))$ (a is dense).

The second condition can be stated as follows: For any $p \in P$, the image of $P \downarrow p$ under a is dense below $a(p)$.
If there is such projection we say that P is completely projected onto Q. Note that even if a is not surjective, the density condition implies that the image of P under a is dense in Q, hence forcing isomorphic to Q.
The following lemma shows that up to forcing isomorphism, complete projections and complete embeddings are in fact the same. That is, if P completely projects onto Q, then Q completely embeds into a poset which is forcing isomorphic to P.

Lemma 0.7. *Suppose $a\colon P \longrightarrow Q$ is a complete projection. Define a Q-name for a poset $\dot{R} = \left\{p \in P;\, a(p) \in \dot{H}\right\}$, where $\dot{H}$ is the Q-name for a Q-generic. Define $i : P \longrightarrow Q * \dot{R}$ by*

$$i(p) = (a(p), p).$$

Then i is a dense embedding.

Proof. (1) of definition 0.5 is clear as a is a homomorphism. (2) holds since if

$$(a(p), p) = i(p) \parallel i(q) = (a(q), q),$$

then in particular $p \parallel q$.
For (3), we show that $\{(a(p), p);\, p \in P\}$ is dense in $Q * \dot{R}$.
Take a condition $(q, \dot{r}) \in Q * \dot{R}$. Then by definition $q \Vdash \dot{r} \in R$, i.e. $q \Vdash \dot{r} \in P \wedge a(\dot{r}) \in \dot{H}$. Thus there is $s \leq q$ and $t \in P$ s.t $s \Vdash t = \dot{r}$. In particular $s \Vdash a(t) \in \dot{H}$, hence, by separativity, $s \leq a(t)$.
Now by the density condition on a, there is some $p \leq t$ s.t $a(p) \leq s$. Hence $(a(p), p) \leq (s, t) \leq (q, \dot{r})$. $\square$

By the above lemma and the standard facts on dense embeddings, we have in particular that if $G \subset P$ is P-generic over V, then $H = a''G$ is Q-generic over V and G is $\dot{R}^H$-generic over $V[H]$. Furthermore, if H is a Q-generic over V, G a $\dot{R}^H$-generic over $V[H]$, then G is also P-generic over V.

The following facts are commonly used for showing the distributivity or strategic closure of forcing notions.

Fact. *Suppose P, Q are posets and Q is completely embedded into P. Then*

$$P \text{ is } \kappa\text{-distributive} \implies Q \text{ is } \kappa\text{-distributive}.$$

The proof is simple, as any Q-generic extension can be embedded in a P-generic extension, hence cannot add new subsets of ordinals of size $< \kappa$.

Fact. *Suppose P,Q are posets and either there is a complete projection of P onto Q or a complete embedding of Q into P. Then*

$$P \text{ is } \kappa\text{-strategically closed} \implies Q \text{ is } \kappa\text{-strategically closed}.$$

A proof of the fact above is included in the appendix, see lemma 5.2.
Notice that distributivity is a property invariant under forcing equivalence, unlike closure and strategic closure, which are properties of the partial order. That is, for a poset P, we may be interested in finding a forcing equivalent poset which is closed, and not only in whether P itself satisfies closure properties.

0.2. **Ultrafilters and ultrapowers.** We review here some basic definitions and facts regarding ultrafilters, ultrapowers and ultrapower embeddings. We refer to [4] and [5] for a thorough treatment.

Let X be a set. We say that $\mathscr{U}$ is an ultrafilter over X if $\mathscr{U}$ is a filter on $P(X) = \{Y; Y \subset X\}$ and for any $Y \subset X$, either $Y \in \mathscr{U}$ or $X \setminus Y \in \mathscr{U}$. For a subset $A \subset X$, A is said to be *large* iff $A \in \mathscr{U}$.
Given a (set) model M for the language $\mathscr{L}$ we can form the ultrapower of M by $\mathscr{U}$, defined by

$$\mathrm{Ult}(M,\mathscr{U}) = \{[f]_{\mathscr{U}}; f : X \longrightarrow M\},$$

where $[f]_{\mathscr{U}}$ is the equivalence class of f under the equivalence relation $=_{\mathscr{U}}$ on $\prod_X M$, defined by

$$f =_{\mathscr{U}} g \iff \{x \in X; f(x) = g(x)\} \in \mathscr{U}.$$

The subscript $\mathscr{U}$ may be omitted from $[\ ]_{\mathscr{U}}$ when clear from context.
We make $\mathrm{Ult}(M,\mathscr{U})$ a structure to the language $\mathscr{L}$ as follows. For a predicate symbol $P \in \mathscr{L}$ define the interpretation of P in $\mathrm{Ult}(M,\mathscr{U})$ by

$$[f]_{\mathscr{U}} \in P^{\mathrm{Ult}(M,\mathscr{U})} \iff \left\{x \in X; f(x) \in P^M\right\} \in \mathscr{U}.$$

We define a corresponding embedding $j : M \longrightarrow \mathrm{Ult}(M,\mathscr{U})$ as follows. For $a \in M$, let $f_a : X \longrightarrow M$ be the constant function a, that is $\forall x \in X\, (f_a(x) = a)$. Then define

$$j(a) = [f_a]_{\mathscr{U}}.$$

Łoś's theorem states that for any formula $\varphi(\bar{x})$ for the language $\mathscr{L}$ and $[f]_1, \dots, [f_n] \in \mathrm{Ult}(M,\mathscr{U})$,

$$\mathrm{Ult}(M,\mathscr{U}) \vDash \varphi([f]_1, \dots, [f_n]) \iff \{x \in X; \varphi(f_1(x), \dots, f_n(x))\} \in \mathscr{U}.$$

In particular, for any $\bar{a} \subset M$,

$$\mathrm{Ult}(M,\mathscr{U}) \vDash \varphi(j(\bar{a})) \iff M \vDash \varphi(\bar{a}).$$

That is to say j is an elementary embedding.
Similarly, when $\mathscr{U} \in V$, where V is a set theoretic universe, we can take the ultrapower of V by $\mathscr{U}$ and get $j : V \longrightarrow \mathrm{Ult}(V,\mathscr{U})$ s.t j is definable in V and is an elementary embedding for the language $\in$. See [5] for details on the construction and its definability.

For a cardinal λ, we say that an ultrafilter $\mathscr{U}$ is λ-complete if the intersection of $< \lambda$ many large sets is large. Recall the following fact (see [5]):

$$\mathrm{Ult}(V,\mathscr{U}) \text{ is well founded if and only if } \mathscr{U} \text{ is } \omega_1\text{-complete.}$$

If $\mathscr{U}$ is ω_1-complete, $\mathrm{Ult}(V,\mathscr{U})$ is isomorphic to a transitive class N via the Mostowski collapse. In this case we identify $\mathrm{Ult}(V,\mathscr{U})$ with its Mostowski collapse N, and denote the corresponding embedding by $j : V \longrightarrow N$. Furthermore, for any $f : \lambda \longrightarrow V$ we denote $[f]_{\mathscr{U}}$ as the corresponding element in N.

A cardinal λ is measurable if and only if there is a λ-complete ultrafilter over λ.
Recall the following fact:

Fact. *The following are equivalent:*

- There is a non trivial ω_1-complete ultrafilter.
- There is a transitive class $N \subset V$, $N \neq V$ and an elementary embedding $j : V \longrightarrow N$.
- There exists a measurable cardinal.

For definitions, proofs and a comprehensive discussion on the subject, see [5]. Here we will mention the following fact, which can serve as a definition for a measurable cardinal for our purposes.

Fact. *Suppose λ is a measurable cardinal, then there exists an ultrafilter $\mathscr{U}$ over λ (a normal ultrafilter), such that if N is the transitive collapse of* $\mathrm{Ult}(V, \mathscr{U})$*, j the corresponding ultrapower embedding, i.e.*

$$j : V \longrightarrow N \simeq \mathrm{Ult}(V, \mathscr{U}),$$

then we have:

- *N is closed under λ-sequences, i.e. $N^{\lambda} \subset N$.*
- *$[id]_{\mathscr{U}} = \lambda$, where $id : \lambda \longrightarrow \lambda$ is the identity function.*
- *$[f]_{\mathscr{U}} = j(f)(\lambda)$ for any $f : \lambda \longrightarrow V$.*

0.3. **Ultrapowers of forcing extensions.** Suppose $\mathscr{U}$ is a normal ultrafilter over a measurable cardinal λ with a corresponding ultrapower embedding

$$j : V \longrightarrow N \simeq \mathrm{Ult}(V, \mathscr{U}).$$

Suppose P is a λ^+-distributive forcing notion and G a P-generic over V. By distributivity, no new subsets of λ are added in $V[G]$, hence in $V[G]$, $\mathscr{U}$ is an ultrafilter over λ. We can therefore take the ultrapower of $V[G]$ by $\mathscr{U}$ and we have a corresponding embedding

$$\tilde{j} : V[G] \longrightarrow \tilde{N} \simeq \mathrm{Ult}(V[G], \mathscr{U}).$$

Claim. For any $f \in V$, $f : \lambda \longrightarrow V$ we have $[f]^{V}_{\mathscr{U}} = [f]^{V[G]}_{\mathscr{U}}$.

Proof. We prove by induction on the well founded relation $\in$. Let f be as in the claim and suppose for any $g \in V$, $g : \lambda \longrightarrow V$ s.t $[g]^V \in [f]^V$ we have $[g]^V = [g]^{V[G]}$.
If $[g]^V \in [f]^V$ then clearly $[g]^V = [g]^{V[G]} \in [f]^{V[G]}$.
If $[g]^{V[G]} \in [f]^{V[G]}$, then w.l.o.g $\forall \eta < \lambda\, (g(\eta) \in f(\eta))$, in particular $g : \lambda \longrightarrow V$. By λ^+-distributivity we have $g \in V$, hence by induction hypothesis $[g]^{V[G]} = [g]^V \in [f]^V$. ☐

The claim above in particular shows that $\tilde{j}$ extends j, i.e. $\tilde{j}\restriction_V = j$. Furthermore, we claim that $\tilde{j}(G)$ is $\tilde{j}(P)$-generic over N and $\tilde{N} = N\left[\tilde{j}(G)\right]$.

To simplify the definability of V in the generic extension we note the following. When constructing a generic extension of V, we can do so while adding a predicate $\check{V}$ for the language of set theory. The forcing theorem shows that the generic extension is a model of the appropriate ZFC axioms with the new predicate (and of course, $\tau \in V$ iff $\tau = \check{x}$ for some $x \in V$). When taking an ultrapower of the generic extension, we therefore get an elementary embedding with respect to the language $\left(\in, \check{V}\right)$. Now we have that $V[G]$ thinks it is a generic extension of V with a generic G. Hence by elementarity, $\tilde{N}$ thinks it is a generic extension of $\check{V}^{\tilde{N}}$ with a generic $\tilde{j}(G)$. Thus it only remains to show that $\check{V}^{\tilde{N}} = N$. This again follows from distributivity. Suppose $f \in V[G]$, $f\colon \lambda \longrightarrow V[G]$ s.t $[f] \in \check{V}^{\tilde{N}}$, then w.l.o.g we can assume $f\colon \lambda \longrightarrow \check{V}^{V[G]} = V$. Hence by λ^+-distributivity, $f \in V$, thus $[f] \in N$.

To see that λ^+-distributivity is essential, we include in the appendix an example for a forcing notion P, s.t no new subsets of λ are added after forcing with P (hence $\mathscr{U}$ remains an ultrafilter), yet the conclusions above fail. See section 5.1.

1. Forcing with $j(P)$

We fix a measurable cardinal λ, a normal ultrafilter $\mathscr{U}$ over λ, and the corresponding elementary embedding

$$j : V \longrightarrow N \simeq \mathrm{Ult}(V, \mathscr{U}).$$

Let P be a forcing notion. We study the ultrapower of P, $j(P)$, as a forcing notion in V.

Remark. The measurability of λ is not essential in most of the arguments, yet we choose to enjoy the structural and notational luxuries of ultrapower embeddings by normal ultrafilters. There will be some exceptions, where we will use the fact that $N^\lambda \subset N$, and we emphasize this when we do (mostly in section 4). Also, the fact that we study an ultrapower of a given poset P is not essential. Analogous claims and proofs can be given for any forcing notion Q in N (which is then represented as $Q = \prod_{\alpha<\lambda} P_\alpha/\mathscr{U}$, for some posets $\{P_\alpha; \alpha < \lambda\}$).

First we show how to embed $j(P)$ in $P^\lambda = \prod_\lambda P$, the product of λ copies of P. Consider the map $[\,]_{\mathscr{U}} : P^\lambda \longrightarrow j(P)$, defined by $f \mapsto [f]_{\mathscr{U}}$.

Lemma 1.1. *The map $[\,]_{\mathscr{U}} : P^\lambda \longrightarrow j(P)$ is a complete projection.*

Proof. Recall definition 0.6 of a complete embedding. $[\,]_{\mathscr{U}}$ is clearly a homomorphism, as

$$f \leq_{P^\lambda} g \implies \forall \eta < \lambda\, (f(\eta) \leq_P g(\eta)) \implies [f]_{\mathscr{U}} \leq_{j(P)} [g]_{\mathscr{U}}.$$

To see that $[\,]_{\mathscr{U}}$ is dense, take $f \in P^\lambda$ and $p \in j(P)$ s.t $p \leq_{j(P)} [f]_{\mathscr{U}}$. We must find a $g \in P^\lambda$ s.t $g \leq_{P^\lambda} f$ and $[g]_{\mathscr{U}} \leq_{j(P)} p$. Take any $\tilde{g}$ s.t $[\tilde{g}]_{\mathscr{U}} = p$. Since $p \leq_{j(P)} [f]_{\mathscr{U}}$, we have $\{\eta; \tilde{g}(\eta) \leq_P f(\eta)\} \in \mathscr{U}$. Thus, by changing $\tilde{g}$ on a small set we get a $g \in P^\lambda$ s.t $[g]_{\mathscr{U}} = [\tilde{g}]_{\mathscr{U}}$ and $\forall \eta < \lambda\, (g(\eta) \leq_P f(\eta))$. Hence $g \leq_{P^\lambda} f$ and $[g]_{\mathscr{U}} = p$, as required. □

Recall section 0.1.3. It follows from lemma 1.1 that $j(P)$ embeds into P^λ, as forcing notions. More specifically, we can define a $j(P)$-name for a poset $\dot{R}$, s.t for any generic $H \subset j(P)$,

$$\dot{R}^H = \left\{ f \in P^\lambda; [f]_{\mathscr{U}} \in H \right\}.$$

There is a dense embedding $i\colon P^\lambda \longrightarrow j(P) * \dot{R}$ defined by $i(f) = ([f]_{\mathscr{U}}, f)$. The following lemma is a trivial corollary of the simple facts on dense embeddings, applied to i. We include a proof for completeness.

Lemma 1.2. *Suppose G is P^{λ}-generic over V. Define*

$$H = \{[f]_{\mathscr{U}}\,; f \in G\}\,.$$

Then H is $j(P)$-generic over V.

Proof. Suppose $V \ni D \subset j(P)$ is a dense subset of $j(P)$. We wish to show that $H \cap D \neq \emptyset$. Define $\tilde{D} = \left\{f \in P^{\lambda}; [f]_{\mathscr{U}} \in D\right\}$, then $\tilde{D}$ is dense in P^{λ}:
For any $f \in P^{\lambda}$, let $p = [f]_{\mathscr{U}} \in j(P)$. By density of D in $j(P)$ we have some $p \geq q \in D$. Take $g \in P^{\lambda}$ s.t $g \leq f$ and $[g]_{\mathscr{U}} = q$, then $g \in \tilde{D}$.
Now, by genericity of G we have some $h \in G \cap \tilde{D}$, thus $[h]_{\mathscr{U}} \in H \cap D$. ☐

Furthermore, any $j(P)$-generic extension can be extended, by forcing with $\dot{R}$, to a P^{λ}-generic extension. Thus any $j(P)$-generic over V can be thought of as being derived as in lemma 1.2.

Corollary 1.3. *If P^{λ} is κ-distributive, then $j(P)$ is κ-distributive.*

Proof. By the discussion above, any $j(P)$-generic extension can be embedded into a P^{λ}-generic extension. Thus if P^{λ} adds no subsets of ordinals of size $< \kappa$, neither does $j(P)$. ☐

In particular, if P is κ-closed, P^{λ} is κ-closed as well, hence $j(P)$ is κ-distributive by the above corollary. Note that in any case, if P is λ-closed, then by elementarity, $N \vDash j(P)$ is $j(\lambda)$-closed, in particular $N \vDash j_{\lambda}(P)$ is λ^{+}-closed. Since $N^{\lambda} \subset N$, we get that $V \vDash j_{\lambda}(P)$ is λ^{+}-closed. Also note that for $\kappa > \lambda^{+}$, $j(P)$ will typically not be κ-closed as $j(P) \subset N$ and N is not closed under λ^{+}-sequences.

On the other hand, for strategic closure we have:

Corollary 1.4. *If P is κ-strategically closed then $j(P)$ is κ-strategically closed.*

Proof. First note that if P is κ-strategically closed, then so is P^{λ}, as we can use the strategy separately at each coordinate. Now the corollary follows from lemma 1.1, as strategic closure is preserved under complete projections (see lemma 5.2 in the appendix for a proof). ☐

One can ask whether the distributivity of P in general suffices to deduce the distributivity of $j(P)$ as well. In section 2 we show that this is not the case.

Recall section 0.3: Suppose G is Q-generic, where Q is a λ^{+}-distributive forcing notion. Then j can be extended to

$$\tilde{j}: V[G] \longrightarrow N\left[\tilde{j}(G)\right] \simeq \mathrm{Ult}(V[G], \mathscr{U}),$$

where N is the ultrapower of V as computed by $V[G]$ as well. In this case we abuse notation and note $\tilde{j}$ by j.
Assume $j(P)$ is λ^{+}-distributive, then we have $j\colon V[H] \longrightarrow N[j(H)]$. Note that H is in particular $j(P)$-generic over N, as $j(P) \in N$. Also, by elementarity, $j(H)$ is $j^{2}(P)$-generic over N (where $j^{2} = j \circ j$). In Corollary 1.6 below we show that H and $j(H)$ are in fact mutually generic over N.

Lemma 1.5. *Suppose G is P^{λ}-generic, $H = \{[f]_{\mathscr{U}}\,; f \in G\}$ is the corresponding $j(P)$-generic. For $\alpha < \lambda$, let $G_{\alpha} = \{f(\alpha)\,; f \in G\}$ be the projection of G to the αth coordinate. Then for any $\alpha_0 < \lambda$, G_{α_0} and H are mutually generic over V.*

Proof. W.l.o.g $\alpha_0 = 0$. The argument here is carried out in the forcing extension $V[G]$.
By fact 0.3 about mutual genericity, it suffices to show that G_0 is P-generic over $V[H]$. Suppose not, then there is $f \in P^\lambda$ and $\dot{D}$ a $j(P)$-name for a dense subset of P s.t $f \Vdash \dot{D}^{\dot{H}} \cap G_0 = \emptyset$.
Define $p = [f] \in j(P)$. As $\dot{D}$ is a $j(P)$-name for a dense subset of P, there is a $q \in P$, $q \leq_P f(0)$ and $p' \in j(P)$, $p' \leq_{j(P)} p$ s.t $p' \Vdash_{j(P)} q \in \dot{D}$.
Take $\tilde{g} \in P^\lambda$ s.t $[\tilde{g}] = p'$ and $\tilde{g} \leq_{P^\lambda} f$. Define

$$g(\alpha) = \begin{cases} q & \alpha = 0 \\ \tilde{g}(\alpha) & \alpha \neq 0 \end{cases}$$

$q \leq f(0)$, hence $g \leq_{P^\lambda} f$. Also, $g(0) = q$ implies that $g \Vdash_{P^\lambda} q \in G_0$. Furthermore, $g \Vdash_{P^\lambda} [g] \in \dot{H}$, and

$$[g] = [\tilde{g}] = p' \Vdash_{j(P)} q \in \dot{D}.$$

Hence g is an extension of f, forcing that $\dot{D}^{\dot{H}} \cap G_0 \ni q$, in contradiction. $\square$

Corollary 1.6. *Assume P^λ is λ^+-distributive. For any H which is $j(P)$-generic over V, in $V[H]$ we have*

$$H \text{ and } j(H) \text{ are mutually generic over } N.$$

Proof. Let G be a P^λ-generic s.t $H = \{[f]_{\mathscr{U}}\,; f \in G\}$. By λ^+-distributivity of P^λ, we have an extension of j to $j\colon V[G] \longrightarrow N[j(G)]$. Let $\{G_\alpha; \alpha < \lambda\}$ be the projections of G, and denote $j(\langle G_\alpha; \alpha < \lambda\rangle) = \left\langle \tilde{G}_\alpha; \alpha < j(\lambda)\right\rangle$. That is, for $\alpha < j(\lambda)$, $\tilde{G}_\alpha = \{f(\alpha)\,; f \in j(G)\}$ is the αth projection of $j(G)$. Note that $G = \prod_{\alpha<\lambda} G_\alpha$, hence

$$H = [\langle G_\alpha; \alpha < \lambda\rangle]_{\mathscr{U}} = j(\langle G_\alpha\rangle; \alpha < \lambda)(\lambda) = \tilde{G}_\lambda.$$

Now, applying j to lemma 1.5 we get that in $N[j(G)]$:

$$\text{For any } \alpha < j(\lambda),\ \tilde{G}_\alpha \text{ and } j(H) \text{ are mutually generic over } N.$$

In particular, for $\alpha = \lambda$, $\tilde{G}_\lambda = H$ and $j(H)$ are mutually generic over N. $\square$

This corollary will be used in section 3.

A bit more about mutual genericity: Fix a $j(P)$-generic H, we work in $V[H]$. Let

$$j^*\colon N \longrightarrow N^* \simeq \mathrm{Ult}(N, j(\mathscr{U}))$$

be the ultrapower embedding of N by $j(\mathscr{U})$, where in N, $j(\mathscr{U})$ is a normal ultrafilter over $j(\lambda)$. Note that for any P which is λ^+-distributive, $N \vDash j(P)$ is $j(\lambda)^+$-distributive. Hence by section 0.3 (applied in N), j^* can be extended to an ultrapower embedding

$$j^*\colon N[H] \longrightarrow N^*[j^*(H)],$$

and $j^*(H)$ is $j^*(j(P))$-generic over N^*.
Also, if $j(P)$ is λ^+-distributive (in V), there is an extension of j,

$$j\colon V[H] \longrightarrow N[j(H)],$$

where $j(H)$ is $j^2(P)$-generic over N.

Claim. $j^*(j(P)) = j^2(P)$.

Proof. j^* is the ultrapower embedding by $j(\mathscr{U})$ as defined in N, so by elementarity, $j^* = j(j)$. Hence for any $x \in V$,

$$j^*(j(x)) = j(j)(j(x)) = j(j(x)) = j^2(x).$$

$\square$

Therefore $j(H)$ and $j^*(H)$ are both $j^2(P)$-generic over N^*.

Proposition 1.7. *Assume P^λ is λ^+-distributive. Then $j(H)$ and $j^*(H)$ are mutually generic over N^*.*

Proof. We work again in the P^λ-generic extension. Take a P^λ-generic G s.t $H = \{[f]_{\mathscr{U}}\,; f \in G\}$. Let $\{G(\alpha)\,; \alpha < \lambda\}$ be the projections, $G = \prod_{\alpha<\lambda} G(\alpha)$. As we've seen above, $H = j(G)(\lambda)$, so by elementarity,

$$j(H) = j^2(G)(j(\lambda)).$$

Similarly, $j(G)$ is $j\left(P^\lambda\right)$-generic over N, and we can extend j^* to $j^* \colon N[j(G)] \longrightarrow N^*[j^*(j(G))]$, where $j^*(j(G)) = j^2(G)$. By elementarity,

$$j^*(H) = j^*(j(G))(j^*(\lambda)) = j^2(G)(\lambda).$$

Now, in $N^*\left[j^2(G)\right]$ we have

$$\forall \alpha < \beta < j^2(\lambda)\left(j^2(G)(\alpha) \text{ and } j^2(G)(\beta) \text{ are mutually generic over } N^*\right).$$

In particular, taking $\alpha = \lambda$ and $\beta = j(\lambda)$, we get that $j(H)$ and $j^*(H)$ are mutually generic over N^*. $\square$

Note that, while the power P^λ produces λ many P-generics over V, the ultrapower $j(P)$ produces only one $j(P)$-generic over N (at least intuitively). Working in a $j(P)$-generic extension, the proposition above shows that we can "resurrect" one of the many P-generics added by P^λ, in the form of a $j^2(P)$-generic over N^*. This idea will motivate the proof of proposition 2.3 below.
Also note that the ultrapower embedding can be further iterated, and in a similar way, we can get more than two mutual generics, over the corresponding iterated ultrapower model.

2. A COUNTER EXAMPLE FOR DISTRIBUTIVITY

In section 1 we have seen (corollary 1.3) that distributivity of P^λ implies distributivity of $j(P)$. In particular, closure of P implies distributivity of $j(P)$.
In this section we show that the mere distributivity of P does not imply the distributivity of $j(P)$. More specifically, for any regular cardinal $\kappa \geq \lambda^+$ we have a forcing notion P which is κ-distributive yet $j(P)$ is not λ^+-distributive.
Note that by corollary 1.3 such a P must in particular be an example to a forcing which is κ-distributive, yet P^λ is not κ-distributive. A known example for this is the forcing notion for threading a $\square$ sequence. We first introduce the definitions and sketch some facts. Recall,

Definition 2.1. Let κ be a cardinal. A $\square(\kappa)$-sequence is a sequence $\mathscr{C} = \langle C_\alpha;\, \alpha < \kappa,\, \alpha \text{ is a limit ordinal}\rangle$ such that:

(1) $C_\alpha \subset \alpha$ is a club.
(2) If $\alpha \in C_\beta$ is a limit ordinal, then $C_\beta \cap \alpha = C_\alpha$.

(3) There is no club $D \subset \kappa$ such that $\forall \alpha \in D\,(D \cap \alpha = C_\alpha)$.

A sequence $\mathscr{C}$ satisfying (1) and (2) is said to be *coherent*. A club $D \subset \kappa$ satisfying $\forall \alpha \in D\,(D \cap \alpha = C_\alpha)$ is said to *thread* the sequence $\mathscr{C}$. Thus a $\square(\kappa)$-sequence is a coherent sequence with no thread. Recall the following forcing notions for adding and destroying a $\square(\kappa)$ sequence.
First, the forcing notion $S = S_\kappa$ for adding a $\square(\kappa)$ sequence. Conditions in S are p s.t:

(1) $p = \langle p_\alpha;\, \alpha \leq \gamma \wedge \alpha \text{ is a limit ordinal}\rangle$ for some $\gamma < \kappa$.
(2) For each $\alpha \leq \gamma$, $p_\alpha \subset \alpha$ is a club in α.
(3) For limit ordinals $\alpha, \beta \leq \gamma$, $\alpha \in p_\beta \implies p_\beta \cap \alpha = p_\alpha$.

The conditions of S are ordered by end extension. Let $\dot{\mathscr{C}} = \left\langle \dot{C}_\alpha;\, \alpha < \kappa,\, \alpha \text{ is a limit ordinal}\right\rangle$ be the name for the S-generic sequence. Define an S-name for a forcing notion, $\dot{T} = T_{\dot{\mathscr{C}}}$, for adding a thread through $\dot{\mathscr{C}}$. Conditions in $\dot{T}$ are limit ordinals below κ, and are ordered by (for $\alpha > \beta$)

$$\alpha <_T \beta \iff \beta \in C_\alpha.$$

Claim. Define the following subset of $S * \dot{T}$:

$$E = \{(p,\gamma)\,;\, p = \langle p_\alpha;\, \alpha \leq \gamma\rangle \in S \wedge \gamma \text{ is a limit ordinal}\}\,.$$

Then E is a κ-closed dense subset of $S * \dot{T}$.

Proof. (Sketch) E is κ-closed: Suppose $\eta < \kappa$ and $\langle (p_\nu, \gamma_\nu)\,;\, \nu < \eta\rangle$ is a descending sequence of elements from E. Take

$$\gamma = \sup_{\nu<\eta} \gamma_\nu,\; q = \bigcup_{\nu<\eta} p_{\gamma_\nu},\; p = \left(\bigcup_{\nu<\eta} p_\nu\right) \frown q.$$

Then (p,γ) is a lower bound of the sequence $\langle (p_\nu,\gamma_\nu)\,;\, \nu < \eta\rangle$.
$E \subset S * \dot{T}$ is dense: Take any condition $(p,\zeta) \in S * \dot{T}$, $p = \langle p_\alpha;\, \alpha \leq \beta\rangle$, and assume w.l.o.g that $\beta \geq \zeta$. Define

$$\tilde{p}_{\beta+\omega} = p_\zeta \cup \{\zeta\} \cup (\beta+\omega \setminus \beta+1)\,, \quad \tilde{p} = p \frown \tilde{p}_{\beta+\omega}.$$

We see that $\tilde{p} \in S$, and by definition, $\tilde{p} \leq p$. Furthermore, $\tilde{p} \Vdash (\beta+\omega) \leq_T \zeta$, as $\zeta \in \tilde{p}_{\beta+\omega}$. Thus $(\tilde{p}, \beta+\omega)$ is in E and extends (p,ζ). $\square$

Thus $S * \dot{T}$ is forcing isomorphic to a κ-closed forcing. In particular, S is κ-strategically closed and $\dot{T}$ is κ-distributive. We can also use the κ-closed dense subset to prove the following fact.

Fact. *If $\mathscr{C}$ is S-generic over V, G is $T_{\mathscr{C}}$-generic over $V[\mathscr{C}]$ and $D = \bigcup_{\alpha\in G} C_\alpha$. Then D is a club in κ, threading $\mathscr{C}$.*

Proof. (Sketch) $V[\mathscr{C}][G]$ is equivalent to an E-generic extension, and if $\tilde{G} \subset E$ is the corresponding E-generic filter, we see that

$$D = \left\{\gamma;\, \exists r \in \tilde{G},\, r = (p,\gamma)\right\}.$$

It follows that D is unbounded in κ, by the κ-closure of E. Also, by definition of D, it satisfies $\forall \alpha \in D\,(D \cap \alpha = C_\alpha)$, hence D is closed, and threads $\mathscr{C}$. $\square$

Assume henceforth that $\mathscr{C} \in V$ was added by forcing with S, hence $T = T_{\mathscr{C}}$ is a κ-distributive forcing notion. The following known fact shows that distributivity is not preserved by products.

Claim 2.2. $T\times T$ is not ω_1-distributive.

Proof. Suppose $D\times E$ is a $T\times T$-generic. Then both D and E thread $\langle C_\alpha;\alpha<\kappa\rangle$.
Suppose by contradiction that in the generic extension $\operatorname{cf}(\kappa)>\omega$.
By genericity, we must have that D and E are different. Take some α on which D and E disagree.
Since $\operatorname{cf}(\kappa)>\omega$, $D,E\subset\kappa$ are clubs, there is a limit ordinal β above α s.t $\beta\in D\cap E$.
Then $D\cap\beta=C_\beta=E\cap\beta$. In contradiction. □

A similar argument can be used to show the following fact:

Fact. *If $\mathscr{C}=\langle C_\alpha;\alpha<\kappa$, α is a limit ordinal$\rangle$ is S-generic over V, then in $V[\mathscr{C}]$, $\mathscr{C}$ is indeed a $\square(\kappa)$-sequence.*

Proof. By the definition of a $\square(\kappa)$-sequence and the definition of S, it only remains to show that $\mathscr{C}$ has no thread in $V[\mathscr{C}]$. Assume otherwise, and let $D\in V[\mathscr{C}]$ be a thread. Let E be a T-generic over $V[\mathscr{C}]$. $V[\mathscr{C}][E]$ is a $S*\dot{T}$-generic extension of V, and as we have seen above, $S*\dot{T}$ is forcing isomorphic to a κ-closed forcing notion. Thus $V[\mathscr{C}][E]\vDash\kappa$ is regular. However, since E is generic over $V[\mathscr{C}]$, we must have that $E\neq D$, and both thread $\mathscr{C}$. This leads to a contradiction, as in claim 2.2. □

Finally, we show that distributivity is not preserved by ultrapowers.

Proposition 2.3. *$j(T)$ is not λ^+-distributive.*

Proof. The idea is, under the assumption of λ^+-distributivity, to construct two mutual generics for $j^2(T)$, as in proposition 1.7. Then we get a contradiction as in claim 2.2. However, we have to be more cautious, as we do not have the assumption that T^λ is λ^+-distributive, which we used in proposition 1.7.
Assume by contradiction that $j(T)$ is λ^+-distributive. Let D be $j(T)$-generic over V.
By λ^+-distributivity we have an extension of j, $j:V[D]\longrightarrow N[j(D)]$ (recall section 0.3). By elementarity we have

$$j(D)\text{ is }j^2(T)\text{-generic over }N.$$

Let $j^*:N\longrightarrow N^*\simeq\operatorname{Ult}(N,j(\mathscr{U}))$ be the ultrapower embedding of N by $j(\mathscr{U})$. As T is κ-distributive, $\kappa\geq\lambda^+$, there is an extension of j^*, $j^*:N[D]\longrightarrow N^*[j^*(D)]$, and by elementarity,

$$j^*(D)\text{ is }j^*(j(T))\text{-generic over }N^*.$$

As in the discussion before proposition 1.7, $j^*(j(T))=j^2(T)$, and $j(D)$, $j^*(D)$ are both $j^2(T)$-generics over N^*. That is, both $j(D)$ and $j^*(D)$ thread $j^2(\mathscr{C})$. To get a contradiction, we show that $j(D)$, $j^*(D)$ are different.
We claim that for any ordinal η there is a limit ordinal $\alpha<\kappa$ s.t the first ordinal of C_α is $\eta+1$, i.e. $C_\alpha\cap\eta+2=\{\eta+1\}$. This is true since $\mathscr{C}$ was introduced by forcing. Consider the poset S for adding a $\square(\kappa)$-sequence. For any condition $p=\langle p_\alpha;\alpha\leq\gamma\wedge\alpha$ is a limit ordinal$\rangle\in S$ (w.l.o.g assume $\gamma>\eta$), we can define $p'=p\cup(\gamma+\omega,p_{\gamma+\omega})$, where $p_{\gamma+\omega}=\{\eta+1\}\cup(\gamma+\omega)\setminus(\gamma+1)$. Then $p'\leq p$ and

$$p'\Vdash_S C_{\gamma+\omega}\cap\eta+2=\{\eta+1\}.$$

Now, working in N, we know that $j(\mathscr{C})$ was introduced by forcing over some ground model. Hence taking $\eta=\lambda$, there is some limit ordinal $\alpha<\kappa$ s.t the first ordinal in $j(\mathscr{C})(\alpha)$ is $\lambda+1$. Consider the condition $\alpha\in j(T)$. We have

$$\alpha\Vdash_{j(T)}D\cap\lambda+2=j(\mathscr{C})(\alpha)\cap\lambda+2=\{\lambda+1\}.$$

W.l.o.g we can assume $\alpha \in D$ (we take D which is $T \downarrow \alpha$-generic), hence $D \cap \lambda + 2 = \{\lambda + 1\}$. In N, the critical point of j^* is $j(\lambda)$, which is greater than λ, so we have

$$j(D) \cap j(\lambda) + 2 = \{j(\lambda) + 1\} \wedge j^*(D) \cap \lambda + 2 = \{\lambda + 1\}.$$

Note that since $V[D]$ thinks $j(D), j^*(D)$ are clubs in the sense of N^*, then $j(D), j^*(D)$ are indeed clubs in κ (in the sense of $V[D]$). Now, as $V[D] \vDash \mathrm{cf}(\kappa) > \omega$, there is a limit ordinal $\beta > j(\lambda)$ s.t $\beta \in j(D) \cap j^*(D)$. Thus we have $\lambda + 1 \in j^*(D) \cap \beta = j^2(\mathscr{C})(\beta) = j(D) \cap \beta$, in contradiction. $\square$

Remark. The important aspects of T that we have used are that it is κ-distributive, and that there cannot be two distinct T-generics in a λ^+-distributive forcing extension. For any other such poset, one can carry the above proof similarly, where proposition 1.7 serves as motivation for the fact that we get different generics by applying j and j^*. We will see another poset satisfying these properties in section 3 below.
Note that the "two" above is not essential. Suppose P is a κ-distributive poset such that there cannot be three (or α, for an ordinal $\alpha < \lambda$) different generics in a λ^+-distributive extension. Similar arguments can show that $j(P)$ is not λ^+-distributive. Such forcing notions, where P^ξ is κ-distributive for $\xi < \alpha$, yet P^α is not λ^+-distributive, exist. For instance, the forcing notion for threading a $\square_{\kappa,<\alpha}$-sequence.

3. Ultrapower of Cohen forcing

The situation we have in mind throughout this section is that κ has some large cardinal property, and we ask questions related to destruction and preservation of this large cardinal property, while forcing with the ultrapower of $\mathrm{Cohen}(\kappa)$. Recall the following common situation in large cardinal forcing:
A is a poset, designed to add some combinatorial object, which usually contradicts some large cardinal property. D is an A-name for a poset, designed to "destroy" the object added by A. The poset $A * D$ is forcing isomorphic to $\mathrm{Cohen}(\kappa)$, therefore, assuming we started with an indestructible large cardinal, D resurrects the large cardinal property of κ.
Three examples are as follows:

- A adds a κ-Suslin tree T, D adds a branch in T.
- A adds a non reflecting stationary subset $S \subset \kappa$, D adds a club disjoint to S.
- A adds a $\square(\kappa)$-sequence $\mathscr{C}$, D adds a thread through $\mathscr{C}$.

For definitions, more examples, and a comprehensive discussion on the subject, see [2]. $\square(\kappa)$-sequences, and the forcings which add and destroy them, were reviewed above in section 2. The forcing notions for adding and destroying non reflecting stationary subsets are reviewed in section 4 below.

In this section, we describe a similar situation, where A is the ultrapower of $\mathrm{Cohen}(\kappa)$. We define a combinatorial principle: *an unresolvable argument over* κ, which contradicts the weak compactness of κ, and show that forcing with the ultrapower of $\mathrm{Cohen}(\kappa)$ adds an unresolvable argument over κ. We then define and study the corresponding D which *resolves* the argument.

Notation. For a cardinal κ and a subset $a \subset \kappa$, we associate a with its characteristic function χ_a. For subsets $a, b \subset \kappa$, we note the statement $\chi_a \subset \chi_b \vee \chi_b \subset \chi_a$ by $a \parallel b$. This coincides with the forcing definitions if we think of χ_a and χ_b as elements in some Cohen poset.

Definition 3.1. Let $\mathscr{U}$ be a normal ultrafilter over λ, $j : V \longrightarrow N$ the corresponding ultrapower embedding, and let $\kappa \geq \lambda$ be a regular cardinal. We say that $\mathscr{F}$ is a *$\mathscr{U}$-argument over* κ if $\mathscr{F}$ is a family of functions $f : \lambda \longrightarrow V$, satisfying

(1) $\forall f\in\mathscr{F}\left([f]_{\mathscr{U}}\in\mathscr{P}_{\kappa}(\kappa)\right)$.
(2) $\forall f,g\in\mathscr{F}\left([f]_{\mathscr{U}}\parallel[g]_{\mathscr{U}}\right)$. In other words, f and g *are compatible* on a large set (coherence condition).
(3) $\bigcup_{f\in\mathscr{F}}[f]_{\mathscr{U}}$ is cofinal in κ.

For a function $g\colon\lambda\longrightarrow V$, we say that g *resolves* the argument $\mathscr{F}$ if

$$[g]_{\mathscr{U}}=\bigcup_{f\in\mathscr{F}}[f]_{\mathscr{U}}\,.$$

An argument $\mathscr{F}$ is *unresolvable* if there is no function g resolving it.

Remark 3.2. If $\mathscr{F}$ is an unresolvable $\mathscr{U}$-argument over κ, then $A=\bigcup_{f\in\mathscr{F}}[f]_{\mathscr{U}}$ satisfies

$$A\subset\kappa,\ A\notin N,\ \forall\alpha<\kappa\,(A\cap\alpha\in N)\,.$$

Similarly, if we have a set A as above, for each $\alpha<\kappa$ take some $f_\alpha\colon\lambda\longrightarrow V$ s.t $A\cap\alpha=[f_\alpha]_{\mathscr{U}}$. Then $\mathscr{F}=\{f_\alpha;\,\alpha<\kappa\}$ is an unresolvable $\mathscr{U}$-argument over κ.

Remark. The measurability of λ is not essential in definition 3.1. An analogous property can be defined for a general ultrafilter $\mathscr{U}$, and the following theorems will hold as well.

Theorem 3.3. *Let $\mathscr{U}$ be an ultrafilter over λ, $\kappa>\lambda$ cardinals.*

(1) *If κ is weakly compact, then every $\mathscr{U}$-argument over κ is resolvable.*
(2) *If κ is θ-strongly compact, $\theta\geq\kappa$ a cardinal, then every $\mathscr{U}$-argument over θ is resolvable.*

Proof. 1. Recall that for an uncountable cardinal κ, "κ is weakly compact" is equivalent to the following:

$$\forall\alpha<\kappa\left(\kappa\longrightarrow(\kappa)^2_\alpha\right).$$

Let $\mathscr{U}$ be an ultrafilter over a cardinal $\lambda<\kappa$, and let $\mathscr{F}$ be a $\mathscr{U}$-argument over κ. κ is weakly compact, hence inaccessible, so $2^\lambda<\kappa$, and we have $\kappa\longrightarrow(\kappa)^2_{2^\lambda}$. Also, $|\mathscr{P}_\kappa\kappa|=\kappa$ and $\kappa^\lambda=\kappa$, so $\mathscr{F}$ is of size κ. Let $\mathscr{F}=\langle f_\alpha;\,\alpha<\kappa\rangle$ be an enumeration of $\mathscr{F}$, w.l.o.g, assume $[f_\alpha]_{\mathscr{U}}\subset\alpha$. Note that by assumption, $\bigcup_{\alpha<\kappa}[f_\alpha]_{\mathscr{U}}$ is cofinal in κ.
Define the following partition $h:[\kappa]^2\longrightarrow\mathscr{P}(\lambda)$. For any $\alpha<\beta<\kappa$,

$$h(\alpha,\beta)=\left\{\eta<\lambda;\,f_\alpha(\eta)\parallel f_\beta(\eta)\right\}.$$

Note that for any $\alpha<\beta$ we have $[f_\alpha]\parallel\left[f_\beta\right]$, hence $h(\alpha,\beta)\in\mathscr{U}$. Therefore $h\colon\,[\kappa]^2\longrightarrow\mathscr{U}$.
By $\kappa\longrightarrow(\kappa)^2_{2^\lambda}$, there exists a large homogenous subset $X\subset\kappa$. That is, $|X|=\kappa$ and there is some $A\in\mathscr{U}$ s.t

$$\forall\alpha<\beta\,(\alpha,\beta\in X\implies h(\alpha,\beta)=A)\,.$$

Define $g:\lambda\longrightarrow\mathscr{P}(\kappa)$ by

$$\forall\eta<\lambda,\quad g(\eta)=\bigcup_{\alpha\in X}f_\alpha(\eta)\,.$$

By definition of X and A, for any $\alpha\in X$ we have

$$\{\eta<\lambda;\,f_\alpha(\eta)\parallel g(\eta)\}\supset A\in\mathscr{U}\,.$$

Therefore,

$$\forall\alpha\in X\,([f_\alpha]\parallel[g])\,,\text{ hence }\bigcup_{\alpha\in X}[f_\alpha]\parallel[g]\,.$$

Since $|X| = \kappa$, $\bigcup_{\alpha \in X} [f\alpha]$ is cofinal in κ. So $\bigcup_{\alpha \in X} [f\alpha]$ and $[g]$ are both cofinal subsets of κ and are compatible, hence $\bigcup_{\alpha \in X} [f\alpha] = [g]$. Finally, by the coherence of $\mathscr{F}$,

$$\bigcup_{\alpha < \kappa} [f\alpha] = \bigcup_{\alpha \in X} [f\alpha] = [g] .$$

Thus g resolves the argument $\mathscr{F}$, and we are done.

2. Let $j^{\theta} : V \longrightarrow M$ be a θ-strong compactness embedding generated by a fine ultrafilter $\mathscr{V}$ over $P_{\kappa}\theta$.

Remark 3.4. Note that we have $j^{\theta}(\mathscr{U}) = \mathscr{U} \in M$ and we can take the ultrapower of M by $\mathscr{U}$. We can also go the other way around and take the ultrapower of N by $j(\mathscr{V})$. Since j, j^{θ} are definable we can note the corresponding embeddings as $j\left(j^{\theta}\right) : N \longrightarrow M^{N}$ and $j^{\theta}(j) : M \longrightarrow N^{M}$. We claim that the map $[\,]_{\mathscr{V}} : N^{\mathscr{P}_{\kappa}\theta} \longrightarrow M^{N}$ as computed in N is the same as computed in V, and similarly for the map $[\,]_{\mathscr{U}} : M^{\lambda} \longrightarrow N^{M}$. In particular $j\left(j^{\theta}\right) = j^{\theta}|_{N}$ and $j^{\theta}(j) = j|_{M}$, hence $j\left(j^{\theta}\right)$ and $j^{\theta}(j)$ are essentially the same as j^{θ} and j respectively. Furthermore, both embedding commute, that is, $j \circ j^{\theta} = j^{\theta} \circ j$. We refer to section 5.2 for details of this technical yet simple claim.

Let $\mathscr{F}$ be a $\mathscr{U}$-argument over θ, let $A = \bigcup_{f \in \mathscr{F}} [f]_{\mathscr{U}}$. Then

$$V \vDash \forall \alpha < \theta \, (A \cap \alpha \in N) \wedge A \notin N.$$

Define $\beta = \sup j^{\theta\prime\prime}\theta$, then $\beta < j^{\theta}(\theta)$, and

$$M \vDash j^{\theta}(A) \cap \beta \in N^{M}.$$

By the remark above, $N^{M} = M^{N}$. Note that M^{N} is the Mostowski collapse of $\mathrm{Ult}(N, j(\mathscr{V}))$, hence is contained in N. Thus $j^{\theta}(A) \cap \beta \in N$, and $j^{\theta}|_{N}$ is defined in N, since it is the ultrapower embedding of N by $j(\mathscr{V})$. Hence in N we can define

$$\alpha \in A \iff j^{\theta}(\alpha) \in j^{\theta}(A) \cap \beta .$$

So we get that $A \in N$, a contradiction.

□

Fix a normal ultrafilter $\mathscr{U}$ over λ and the corresponding ultrapower embedding

$$j \colon V \longrightarrow N \simeq \mathrm{Ult}(V, \mathscr{U}) .$$

Fix a cardinal $\kappa \geq \lambda^{+}$ and let $P = P_{\kappa} =$ Cohen forcing at κ. P is κ-closed, hence so is P^{λ}. Therefore $j(P)$ is κ-distributive, and corollary 1.6 can be applied. Note that by normality of $\mathscr{U}$, $j(P)$ is a λ^{+}-closed poset. For notational simplicity, assume $j(\kappa) = \kappa$.

Theorem 3.5. *Forcing with $j(P)$ adds an unresolvable $\mathscr{U}$-argument over κ.*

Proof. Let $H \subset j(P)$ be generic over V. Define

$$\mathscr{F} = \left\{ f \in P^{\lambda} ; [f]_{\mathscr{U}} \in H \right\} .$$

For any $f, g \in \mathscr{F}$, $[f]_{\mathscr{U}}, [g]_{\mathscr{U}} \in H$, therefore $[f]_{\mathscr{U}} \parallel [g]_{\mathscr{U}}$, as H is a filter. Also, by genericity, $\bigcup_{f \in \mathscr{F}} [f]_{\mathscr{U}} = \bigcup_{p \in H} p$ is cofinal in $j(\kappa) = \kappa$. Hence in $V[H]$, $\mathscr{F}$ is a $\mathscr{U}$-argument over κ. We show that $\mathscr{F}$ is unresolvable.

Suppose by contradiction that there is a function $g \in V[H]$ which resolves $\mathscr{F}$, that is, $g\colon \lambda \longrightarrow \mathscr{P}(\kappa)$ and

$$[g]_{\mathscr{U}} = \bigcup_{f \in \mathscr{F}} [f]_{\mathscr{U}} .$$

The embedding j can be extended to

$$j\colon V[H] \longrightarrow N[j(H)] = \mathrm{Ult}(V[H], \mathscr{U}),$$

and then $[g]_{\mathscr{U}} \in N[j(H)]$. Note that

$$\bigcup_{p \in H} p = \bigcup_{f \in \mathscr{F}} [f]_{\mathscr{U}} = [g]_{\mathscr{U}} .$$

It follows that H can be defined in $N[j(H)]$, by $H = \{p \in j(P); p \parallel [g]_{\mathscr{U}}\}$, hence $H \in N[j(H)]$. However, by corollary 1.6, H is generic over $N[j(H)]$, in contradiction. □

Corollary 3.6. *In a $j(P)$-generic extension, κ is not weakly compact and there are no compact cardinals between λ and κ.*

Proof. Follows from theorem 3.5 and theorem 3.3. □

Recall from section 1 that we have a complete projection $[\,]_{\mathscr{U}} : P^{\lambda} \twoheadrightarrow j(P)$. Note that in this case P^{λ} is forcing isomorphic to P. Hence the poset $j(P)$ fits into the general scheme described at the beginning of this section; it adds a certain combinatorial object, an unresolvable argument, which contradicts compactness, and it completely embeds into Cohen(κ). We now study the quotient poset $\dot{R} = P^{\lambda}/j(P)$, which resolves the argument added by $j(P)$. $\dot{R}$ is the $j(P)$-name for a poset s.t for any $j(P)$-generic H,

$$\dot{R}^{H} = \left\{ f \in P^{\lambda}; [f]_{\mathscr{U}} \in H \right\},$$

and ordered by $f \leq g \iff f \supset g$ (same order as P^{λ}). We have a dense embedding $i\colon P^{\lambda} \longrightarrow j(P) * \dot{R}$ defined by $i(f) = ([f]_{\mathscr{U}}, f)$ (recall preliminaries section 0.1.3), hence $j(P) * \dot{R}$ is forcing isomorphic to P^{λ}. In particular, $\Vdash_{j(P)} \dot{R}$ is a κ-distributive forcing notion.

Fix a $j(P)$-generic H over some ground model $\tilde{V}$ and let $V = \tilde{V}[H]$. Let $\mathscr{F} = \left\{ f \in P^{\lambda}; [f]_{\mathscr{U}} \in H \right\}$ be the corresponding unresolvable $\mathscr{U}$-argument over κ, and let $R = \dot{R}^{H}$. If $G \subset R$ is generic over V, we can define $g\colon \lambda \longrightarrow \mathscr{P}(\kappa)$ by

$$g = \bigcup_{f \in G} f.$$

Then g resolves $\mathscr{F}$. (Note that, as R is κ-distributive, $\mathscr{U}$ remains an ultrafilter and $\mathscr{F}$ remains a $\mathscr{U}$-argument over κ in $V[G]$.)

Claim 3.7. R is λ-closed, yet not λ^{+}-closed.

Proof. Suppose $\eta < \lambda$ and $\langle f_{\alpha}; \alpha < \eta \rangle$ is an R-descending sequence. Define $f = \bigcup_{\alpha < \eta} f_{\alpha}$. Clearly $f \in P^{\lambda}$ and $\forall \alpha < \eta \, (f \leq f_{\alpha})$. Hence it only remains to show that $f \in R$, that is, $[f] \in H$. By elementarity, and that $\eta < \lambda$, we get that $[f] = \bigcup_{\alpha < \eta} [f_{\alpha}]$, where $\langle [f_{\alpha}]; \alpha < \eta \rangle$ is a $j(P)$-descending chain. Note that in N, $j(P)$ is simply Cohen forcing, and $\forall\alpha<\eta\,([f_\alpha]\in H)$. It is generally true that the least lower bound of a sequence of elements in the generic is also in the generic. Therefore

$$[f]=\bigcup_{\alpha<\eta}[f_\alpha]\in H.$$

Hence $f\in R$, as required.
To see that R is not λ^+-closed, take any $p\in P$ s.t $j(p)\notin H$ (which can be easily found, as for any two $p,p'\in P$ s.t $p\perp p'$, at least one of $j(p)\,,j(p')$ is not in H). For $\alpha<\lambda$ define $f_\alpha\in P^\lambda$ by:

$$\text{for } \eta<\lambda,\quad f_\alpha(\eta)=\begin{cases}p & \eta<\alpha\\ 1 & \eta\geq\alpha\end{cases}.$$

For any $\alpha<\lambda$, $[f_\alpha]=1\in H$, hence $f_\alpha\in R$. However, if $g\in P^\lambda$ is a lower bound of $\langle f_\alpha;\alpha<\lambda\rangle$, then $\forall\eta<\lambda\,(g(\eta)\leq p)$, hence $[g]\leq j(p)$ and therefore $[g]\notin H$. So there is no lower bound of $\langle f_\alpha;\alpha<\lambda\rangle$ in R. □

Lemma 3.8. *Suppose $\mathscr{U}$ is an ultrafilter over λ and $\mathscr{F}$ is an unresolvable $\mathscr{U}$-argument over κ. Let Q be a λ^+-distributive forcing notion and suppose that Q resolves the argument $\mathscr{F}$. Then $Q\times Q$ is not λ^+-distributive.*

Proof. First, by λ^+-distributivity of Q, $\mathscr{F}$ remains a $\mathscr{U}$-argument over κ. Let $\dot g$ be a Q-name s.t

$$Q\Vdash\dot g\colon\lambda\longrightarrow\mathscr{P}(\kappa)\text{ and }\dot g\text{ resolves }\mathscr{F}.$$

Recall that, by λ^+-distributivity, the ultrapower calculation in the generic extension is the same as in V. $\mathscr{F}$ is unresolvable in V, so we must have $Q\Vdash[\dot g]_{\mathscr{U}}\notin N$, therefore $Q\Vdash\{\eta<\lambda;\dot g(\eta)\notin V\}\in\mathscr{U}$. W.l.o.g we can assume $Q\Vdash\forall\eta<\lambda\,(\dot g(\eta)\notin V)$. Assume by contradiction that $Q\times Q$ is λ^+-distributive, and let $G_1\times G_2$ be $Q\times Q$-generic over V. In particular G_1 and G_2 are both Q-generic over V. For $i\in\{1,2\}$, let

$$g_i=\dot g^{G_i}.$$

By mutual genericity of G_1 and G_2, and the fact that $Q\Vdash\forall\eta<\lambda\,(\dot g(\eta)\notin V)$, we must have

$$\forall\eta<\lambda\,(g_1(\eta)\neq g_2(\eta))\,,\text{ thus }[g_1]_{\mathscr{U}}\neq[g_2]_{\mathscr{U}}\,.$$

However, as both g_1 and g_2 resolve the argument $\mathscr{F}$, we have

$$[g_1]_{\mathscr{U}}=\bigcup_{f\in\mathscr{F}}[f]_{\mathscr{U}}=[g_2]_{\mathscr{U}}\,,$$

In contradiction. □

Applying lemma 3.8 to R, we get

Corollary. *$R\times R$ is not λ^+-distributive.*

Note that R is the quotient of two λ^+-closed posets. by the corollary above, R is not forcing isomorphic to any λ^+-closed poset. More generally:

Corollary 3.9. *Suppose $\mathscr{F}$ is an unresolvable $\mathscr{U}$-argument. Then no λ^+-closed forcing notion resolves $\mathscr{F}$.*

Proof. The product of closed posets is closed, so the corollary follows from lemma 3.8. □

Conclusion 3.10. Suppose κ is weakly compact and indestructible to κ-directed closed forcings. After forcing with $j(P)$ we get a model in which the following hold.

(1) There is an unresolvable $\mathscr{U}$-argument $\mathscr{F}$ over κ.
(2) κ is inaccessible, not weakly compact, and there are no strongly compact cardinals between λ and κ.
(3) The forcing notion $R = P^{\lambda}/j(P)$ resolves $\mathscr{F}$ and resurrects the weak compactness of κ. R is κ-distributive, λ-closed and $R \times R$ is not λ^+-distributive.
(4) For any $\mu < \lambda$ and ultrafilter $\mathscr{W}$ over μ, any $\mathscr{W}$-argument over κ is resolvable.

Proof. (1) is theorem 3.5. κ is inaccessible as $j(P)$ is κ-distributive, and the rest of (2) follows from (1) by theorem 3.3. (3) was proved in the discussion above. For (4), suppose by contradiction that for some $\mu < \lambda$, and for some ultrafilter $\mathscr{W}$ over μ, an unresolvable $\mathscr{W}$-argument over κ was introduced by $j(P)$. R is λ-closed, hence by corollary 3.9, forcing with R does not resolve the argument. Thus after forcing with R, we get a $j(P) * R$-generic extension, in which there is an unresolvable $\mathscr{W}$-argument over κ. Thus, by theorem 3.3, κ is not weakly compact in that generic extension. However, $j(P) * R$ is forcing isomorphic to P^{λ}, which is κ-directed closed. Therefore, by our assumption on κ, $j(P) * R \Vdash \kappa$ is weakly compact, in contradiction. □

Question. *As mentioned in the beginning of this section, the situation described above is satisfied by some well known forcing notions, which correspond to combinatorial principles that contradict compactness. It is therefore natural to ask how does the combinatorial principle of having an unresolvable argument interacts with the common combinatorial principles. For instance, does it imply the existence of any of them, or vice versa. More specifically, we can ask which combinatorial principles are added after forcing with $j(P)$.*

Ultrapower of the preparation forcing. For the rest of this section we assume familiarity with supercompact cardinals, reverse Easton forcing and extension of elementary embedding. See [2] as a master reference.

Recall that if κ is supercompact, there is a *preparation forcing* $P_{<\kappa}$ s.t κ remains supercompact after forcing with $P_{<\kappa} * P_{\kappa}$, where here P_{κ} is Cohen forcing as defined in $V^{P_{<\kappa}}$. More generally, we have Laver's indestructibility theorem for supercompact cardinals:

Any supercompact cardinal can be made indestructible under κ-directed closed forcing.

We have seen above that $j(\mathrm{Cohen}(\kappa))$ always destroys weak compactness. Next we show that if we take the ultrapower of the preparation forcing as well, supercompactness is preserved. More explicitly, if κ is supercompact and we force with $j(P_{<\kappa} * P_{\kappa})$, then κ remains supercompact in the generic extension.
We define here $P_{<\kappa}$ as follows. Assume κ is supercompact and $g : \kappa \longrightarrow \kappa$ is a function satisfying

$$\forall \theta \geq \kappa \exists j^{\theta} \left[j^{\theta} : V \longrightarrow M_{\theta} \text{ is a } \theta\text{-supercompactness embedding for } \kappa \wedge j^{\theta}(g)(\kappa) > \theta \right].$$

Define $P_{<\kappa}$ to be the reverse Easton iteration of P_{α}, where P_{α} is Cohen forcing at α if $\lambda < \alpha < \kappa$, $g''\alpha \subset \alpha$ and α is inaccessible, and P_{α} is trivial otherwise. Let $P = P_{<\kappa} * P_{\kappa}$.
We know that in a P-generic extension, for any θ we can extend a θ-supercompactness embedding j^{θ}, hence κ remains supercompact.
Note that P is λ^+-closed, hence $j(P)$ is λ^+-distributive.

Proposition 3.11. *After forcing with $j(P)$, κ remains supercompact.*

Remark. Note that this is not a "preparation forcing for $j(P_\kappa)$" (as would contradict corollary 3.6). While $j(P_{<\kappa} * P_\kappa)$ is trivially decomposed as $j(P_{<\kappa}) * j(P_\kappa)$, this is not the same as forcing with $j(P_{<\kappa})$, and then forcing with the ultrapower of κ-Cohen forcing, as defined in $V^{j(P_{<\kappa})}$.

Proof. First note that for $\lambda < \mu < \kappa$, μ inaccessible, we have a decomposition

$$j(P) = j\left(P_{<\mu} * P_{\geq\mu}\right) = j\left(P_{<\mu}\right) * j\left(P_{\geq\mu}\right).$$

$P_{\geq\mu}$ is μ-closed, hence by corollary 1.3, $j\left(P_{\geq\mu}\right)$ is μ-distributive.
Let $H = H_{<\kappa} * H_\kappa$ be $j(P)$-generic over V. Take $\zeta \geq \kappa$, we wish to show that κ is ζ-supercompact in $V[H]$. W.l.o.g assume that $\zeta^{<\kappa} = \zeta$.
Take $\theta \geq 2^\zeta$ and let $j^\theta : V \longrightarrow M$ be a θ-supercompactness embedding in V s.t $j^\theta(g)(\kappa) > \theta$.
Let μ be the first ordinal above κ such $j^\theta(P)(\mu)$ is not trivial, then $\mu > \theta$. μ is inaccessible in M, hence $j(\mu) = j^\theta(j)(\mu) = \mu$ and μ is the first ordinal such that $j\left(j^\theta(P)\right)(\mu)$ is not trivial (recall remark 3.4). Hence μ is also the first ordinal s.t $j^\theta(j(P))(\mu)$ is not trivial, as $j\left(j^\theta(P)\right) = j^\theta(j(P))$. Thus we have a decomposition in M,

$$j^\theta(j(P)) = \underbrace{\left(j^\theta(j(P))\right)_{<\mu}}_{=j(P)} * j^\theta(j(P))_{\geq\mu},$$

where $j^\theta(j(P))_{\geq\mu}$ is θ^+-distributive in M.
Let $p \in j^\theta(P)_{\geq\mu}$ be a master condition for P. i.e. if $\dot{G}$ is the P-name for a P-generic, then

$$\Vdash_P \forall r \in \dot{G}\left(j^\theta(r) \geq p\right).$$

By elementarity, in N we have

$$\Vdash_{j(P)} \forall r \in j\left(\dot{G}\right)\left(j\left(j^\theta\right)(r)\right) \geq j(p),$$

where $j\left(\dot{G}\right)$ is the $j(P)$-name for a $j(P)$-generic. Hence $j(p)$ is a master condition for $j(P)$. Note that for $r \in j\left(\dot{G}\right) \subset j(P)$, $j\left(j^\theta\right)(r) = j^\theta(r)$.
Now we can apply the usual arguments for constructing a normal measure over $\mathscr{P}_\kappa\zeta$.
For $\dot{t}$ which is a nice $j(P)$-name for a subset of $\mathscr{P}_\kappa\zeta$, consider the dense open set of $j^\theta(j(P)))_{\geq\mu}$,

$$D_{\dot{t}} = \left\{r \in j^\theta(j(P)))_{\geq\mu}; \exists h \in H\left((h \frown r) \parallel j^{\theta\prime\prime}\theta \in j^\theta(\dot{t})\right)\right\},$$

where for $h \in H$, we consider it as an element of $\left(j^\theta(j(P))\right)_{<\mu} = j(P)$. As $\zeta^{<\kappa} = \zeta$, $|\mathscr{P}_\kappa\zeta| = \zeta$. We can think of a nice name for a subset of $\mathscr{P}_\kappa\zeta$ as a function from $P_\kappa\zeta$ to the set of all antichains in $j(P)$. Thus, as $|j(P)| = \kappa$ there are at most $(2^\kappa)^\zeta = 2^\zeta$ such names $\dot{t}$.
By θ^+-distributivity of $j^\theta(j(P))\,|_{\geq\mu}$, we have that $D = \bigcap_{\dot{t}} D_{\dot{t}}$ is dense. Take $q_0 \in D$ s.t $q_0 \leq j(p)$.
Next, for a nice name for a choice function $\dot{f} : \mathscr{P}_\kappa\zeta \longrightarrow \zeta$, and $\alpha < \theta$, define the dense open set

$$D_{\dot{f},\alpha} = \left\{r \in j^\theta(j(P))_{\geq\mu}; \exists h \in H\left((h \frown r) \parallel j^\theta(f)\left(j^{\theta\prime\prime}\theta\right) = j(\alpha)\right)\right\}.$$

As before, for each $\alpha < \theta$ there are 2^{ζ} such nice names $\dot{f}$. Hence we have θ such pairs $\left(\dot{f},\alpha\right)$. By θ^{+}-distributivity, take $q \in j^{\theta}\left(j\left(P\right)\right)$ s.t $q \in \bigcap_{\left(\dot{f},\alpha\right)} D_{\dot{f},\alpha}$ and $q \leq q_0$.
Define a filter $\mathscr{V}$ on $\mathscr{P}_{\kappa}\zeta$ as follows: for $X \in V\left[G\right], X \subset \mathscr{P}_{\kappa}\zeta$, take a nice name $\dot{t}$ s.t $X = \dot{t}^{G}$ and define

$$X \in \mathscr{V} \iff \exists r \in H\left(r \frown q \Vdash j^{\theta\prime\prime}\theta \in j^{\theta}\left(\dot{t}\right)\right).$$

We claim that $\mathscr{V}$ is a normal ultrafilter over $P_{\kappa}\zeta$ in $V\left[H\right]$:
First, $\mathscr{V}$ is well defined since $q \leq j\left(p\right)$ and $j\left(p\right)$ is a master condition.
Furthermore, $\mathscr{V}$ is an ultrafilter since for any nice name $\dot{t}$, there is $h \in H$ s.t $h \frown q$ decides the statement $j^{\theta\prime\prime}\theta \in j^{\theta}\left(\dot{t}\right)$.
To see that $\mathscr{V}$ is normal, take a choice function $f : \mathscr{P}_{\kappa}\zeta \longrightarrow \zeta$ in $V\left[H\right]$. Let $\dot{f}$ be a nice name with $\dot{f}^{H} = f$. Then

$$\Vdash_{j^{\theta}\left(j\left(P\right)\right)} j^{\theta}\left(\dot{f}\right)\left(j^{\theta\prime\prime}\theta\right) \in j^{\theta\prime\prime}\theta.$$

Since $q \in \bigcap_{\alpha<\theta} D_{\dot{f},\alpha}$, we have that for some $\alpha < \theta$ there is $h \in H$ s.t $h \frown q \Vdash j^{\theta}\left(\dot{f}\right)\left(j^{\theta\prime\prime}\theta\right) = j^{\theta}\left(\alpha\right)$.
Hence

$$\left\{t \in \mathscr{P}_{\kappa}\zeta; f\left(t\right) = \alpha\right\} \in \mathscr{V}.$$

□

4. Does $j\left(P\right)$ add a P-generic over V?

When forcing with $j\left(P\right)$ over V, one can ask whether a P-generic is added as well. In this section we give some examples, both for when a P-generic is added and when it is not.
Three trivial examples are as follows. Let $\mathscr{U}$ be a normal ultrafilter over λ and let $P_{\mu} =$ Cohen forcing at μ.
If $P = P_{\mu}$ and $\mu < \lambda$, then $j\left(P\right) = P$.
If $P = P_{\lambda}$, then $j\left(P\right)$ does not add a P-generic over V, since $j\left(P\right)$ is λ^{+}-closed (by normality of $\mathscr{U}$).
If $P = P_{\lambda^{+}}$, assuming GCH, then $j\left(P\right)$ is a λ^{+}-closed forcing of size λ^{+}. Hence $j\left(P\right)$ is forcing isomorphic to P.
In fact, since $j\left(P_{\lambda}\right)$ is also λ^{+}-closed, assuming GCH we have $j\left(P_{\lambda}\right) \simeq j\left(P_{\lambda^{+}}\right) \simeq P_{\lambda^{+}}$ as forcing notions in V.

Theorem 4.1. *Let P be Cohen forcing at κ where $\kappa > \lambda$ is regular. Then forcing with $j\left(P\right)$ over V adds a P-generic.*

Proof. More specifically we show there is a complete projection $a : j\left(P\right) \longrightarrow P$. Thus given a $j\left(P\right)$-generic H, $a''H$ is P-generic.
We define a map $a : j\left(P\right) \longrightarrow P$ as follows. Given $q \in j\left(P\right)$, $q : \mathrm{dom}q \longrightarrow 2$, define $a\left(q\right) \in P$ by

$$a\left(q\right)\left(\alpha\right) = q\left(j\left(\alpha\right)\right).$$

That is, $\alpha \in \mathrm{dom}a\left(q\right) \iff j\left(\alpha\right) \in \mathrm{dom}q$.
First we clam that a is well defined:
Let $q \in j\left(P\right)$, i.e $q \in N$, $q : \mathrm{dom}q \longrightarrow 2$ and $N \vDash \left|\mathrm{dom}q\right| < j\left(\kappa\right)$. Take $f : \lambda \longrightarrow V$ s.t $\left[f\right] = q$ and $\forall\eta < \lambda\left(\left|\mathrm{dom}f\left(\eta\right)\right| < \kappa\right)$. Define $\mu = \sup_{\eta<\lambda}\sup\mathrm{dom}f\left(\eta\right)$. Then $\mu < \kappa$ and $\mathrm{dom}q \subset j\left(\mu\right)$. Thus $\mathrm{dom}\left(a\left(q\right)\right) \subset \mu$, hence $a\left(q\right) \in P$.

Clearly, if $q_1, q_2 \in j(P)$ and $q_1 \leq q_2$ then $a(q_1) \leq a(q_2)$. Hence it only remains to show that a satisfies the density condition.
Take $q \in j(P)$, $p \in P$ s.t $p \leq a(q)$. We find $q' \leq q$ s.t $a(q') = p$. Define q' as follows

$$q'(\alpha) = \begin{cases} q(\alpha) & \alpha \in \mathrm{dom} q \\ j(p)(\alpha) & \alpha \in \mathrm{dom} j(p) \setminus \mathrm{dom} q \end{cases}$$

First of all, $q' \in N$ since q, $j(p) \in N$. Hence $q' \in j(P)$.
Clearly, $q' \leq q$. Also, for any $\alpha \in \mathrm{dom} p$, we have either $j(\alpha) \in \mathrm{dom} q$. Then

$$a\left(q'\right)(\alpha) = q'(j(\alpha)) = q(j(\alpha)) = a(q)(\alpha) = p(\alpha)$$

Otherwise, $j(\alpha) \in \mathrm{dom} j(p) \setminus \mathrm{dom} q$. Then

$$a\left(q'\right)(\alpha) = j(p)(j(\alpha)) = j(p(\alpha)) = p(\alpha).$$

Also note that $\alpha \in \mathrm{dom} p \implies j(\alpha) \in \mathrm{dom} j(p) \implies \alpha \in \mathrm{dom} a(q')$.
Thus $a(q') = p$, and we are done. $\square$

We will see below another class of forcing notions for which P embeds into $j(P)$, as well as a class of forcing notions for which $j(P)$ does not add a P-generic.
First note that the for the trivial example mentioned above, where forcing with $j(P_\lambda)$ adds no P_λ-generic, $(P_\lambda)^\lambda$ is not λ^+-distributive. Hence the results of section 1 do not apply, that is, P_λ is not really "one of the forcing notions we are interested in" for this matter. We consider now a more interesting example in which forcing with $j(P)$ does not add a P-generic. In this case we also have that P^λ is λ^+-distributive.

Take a normal ultrafilter $\mathscr{U}$ over λ and let $\kappa > \lambda$ be inaccessible. We wish to consider the ultrapower of a forcing notion that destroys a stationary subset of κ, and we want such a forcing notion which is also κ-distributive. There are several ways of achieving that, where the most interesting one in our context is construction and destruction of non reflecting stationary subsets. However, we will consider a simpler situation. Let A be Cohen forcing at κ. Given an A-generic $G \subset \kappa$, define

$$S = \{\alpha \in G; \mathrm{cf}(\alpha) = \lambda\}.$$

It is simple to see that S is stationary in $V[G]$: given an A-name for a club subset of κ, we can construct a descending chain of conditions $\langle p_\eta; \eta < \lambda\rangle$ and a strictly ascending sequence of ordinals $\langle \mu_\eta; \eta < \lambda\rangle$ s.t $p_\eta \Vdash \mu_\eta \in \dot{C}$ and $\mu_\eta \leq \sup \mathrm{dom} p_\eta < \mu_{\eta+1}$. Define $\mu = \sup_{\eta<\lambda} \mu_\eta$ and $p = \bigcup_{\eta<\lambda} p_\eta \cup \{(\mu, 1)\}$, then $p \Vdash \mu \in S \cap \dot{C}$.
In $V[G]$, we define the forcing notion $D = D_S$ for adding a club disjoint to S:

$$p \in D \iff p \text{ is a closed bounded subset of } \kappa \text{ and } p \cap S = \emptyset,$$

ordered by reverse inclusion. We associate a D-generic with the club subset of κ it defines.

Claim 4.2. Define a subset $E \subset A * D$,

$$E = \{(p,q); p \in A \wedge q \in D \wedge \mathrm{dom} p = \sup q + 1\}.$$

Then E is a κ-closed dense subset of $A * D$.

Proof. (Sketch) E is κ-closed: Suppose $\mu<\kappa$ and $\langle (p_\nu,q_\nu)\,;\,\nu<\mu\rangle$ is a descending chain of elements in E. Define

$$\zeta=\sup_{\nu<\mu}\operatorname{dom}p_\nu=\sup_{\nu<\mu}\sup q_\nu,\quad p=\bigcup_{\nu<\mu}p_\nu\cup(\zeta,0)\,,\quad q=\bigcup_{\nu<\mu}q_\nu\cup\{\zeta\}\,.$$

Then (p,q) is a lower bound of the sequence $\langle (p_\nu,q_\nu)\,;\,\nu<\mu\rangle$.
E is dense in $A*D$: Take any $(p,q)\in A*D$, w.l.o.g assume that $\operatorname{dom}p>\sup q$. Let $\zeta=\operatorname{dom}p$ and define

$$\tilde q=q\cup(\zeta+\omega+1\setminus\zeta)\,,\quad \tilde p\colon\,\zeta+\omega+1\longrightarrow 2,\,\tilde p(\alpha)=\begin{cases}p(\alpha) & \alpha<\zeta\\ 0 & \zeta\le\alpha\end{cases}.$$

Then $(\tilde p,\tilde q)\in E$ and extends (p,q). $\square$

Thus $A*D$ is forcing isomorphic to a κ-closed forcing notion. In particular, we get that D is forced to be a κ-distributive forcing notion.
Assume henceforth that $V=\tilde V[G]$, where G is A-generic over $\tilde V$. Let $S=\{\alpha\in G;\operatorname{cf}(\alpha)=\lambda\}$ and $D=D_S$. Then D is a κ-distributive forcing notion.

Claim. D is λ-closed.

Proof. Suppose $\eta<\lambda$ and $\langle p_\alpha;\,\alpha<\eta\rangle\subset D$ is a descending chain of conditions. For a set of ordinals q, let $\bar q$ be the closure of q, that is, $q\cup\{\text{limit points of } q\}$. Define $p=\overline{\bigcup_{\alpha<\eta}p_\alpha}$, then p is a closed bounded subset of κ, extending all of p_α. Hence it only remains to show that p is disjoint to S. $\bigcup p_\alpha$ is clearly disjoint to S, and the only new ordinal in $\overline{\bigcup_{\alpha<\eta}p_\alpha}\setminus\bigcup_{\alpha<\eta}p_\alpha$ can be $\sup\bigcup p_\alpha$. By definition we have $\operatorname{cf}(\sup\bigcup p_\alpha)\le\eta<\lambda$, and $\operatorname{cof}(S)=\lambda$, hence $\sup\bigcup p_\alpha\notin S$. $\square$

Fact. *(See lemma 5.5 in the appendix) If κ is inaccessible, $\lambda<\kappa$. $S\subset\kappa$ a stationary subset with $\operatorname{cof}(S)=\lambda$. Then a λ^+-closed forcing notion preserves the stationarity of S, i.e does not add a club disjoint to S.*

D is λ-closed, hence $N\vDash j(D)$ is $j(\lambda)$-closed, in particular $N\vDash j(D)$ is λ^+-closed. Since $N^\lambda\subset N$, we have that $V\vDash j(D)$ is λ^+-closed. Therefore, by the fact above, S remains stationary after forcing with $j(D)$, in particular:

no D-generic is added after forcing with $j(D)$ over V.

Next we show that $\prod_\lambda D$ is κ-distributive.

Claim 4.3. If $S\subset\kappa$ is not stationary, then the poset D_S as defined above is forcing isomorphic to Cohen(κ).

Proof. (Sketch) S is not stationary, so there is a club $E\subset\kappa$ s.t $E\cap S=\emptyset$. Define $\tilde D\subset D$,

$$\tilde D=\{p;\,p\text{ is a closed bounded subset of }\kappa\wedge p\cap S=\emptyset\wedge\sup p\in E\}\,.$$

$\tilde D$ is κ-closed: Suppose $\langle p_\xi;\,\xi<\eta\rangle$ is a descending sequence of conditions in $\tilde D$, $\eta<\kappa$. Then $\sup\bigcup_\xi p_\xi\in E$, as E is closed, therefore $p=\overline{\bigcup_\xi p_\xi}\in\tilde D$ is a lower bound. So $\tilde D$ is a κ-closed forcing notion of size κ, therefore $\tilde D\simeq P_\kappa$.
Furthermore, $\tilde D$ is clearly dense in D (as E is unbounded), hence $D\simeq\tilde D\simeq P_\kappa$. $\square$

Now $\prod_{\lambda} D$ can be decomposed as $D \times \prod_{0<\alpha<\lambda} D$. D is κ-distributive and D is composed of subset of size $< \kappa$, hence

$$V^D \vDash D \text{ is the poset for adding a club disjoint to } S,$$

that is, $D = D_S$ as computed by V^D as well. Also, $V^D \vDash S$ is not stationary, hence in V^D, $D \simeq P_\kappa$. Thus

$$V^D \vDash \prod_{0<\alpha<\lambda} D \simeq \prod_{0<\alpha<\lambda} P_\kappa, \text{ which is } \kappa\text{-closed.}$$

Therefore $D \times \prod_{0<\alpha<\lambda} P_\kappa$ is κ-distributive, hence we get that $\prod_\lambda D$ is κ-distributive.

To complete the discussion on the ultrapower of the forcing notion D_S, we consider forcing with $j(D_S)$, where $S \subset \kappa$ is a stationary subset such that $\{\nu \in S; \mathrm{cf}(\nu) = \lambda\}$ is not stationary. W.l.o.g we can assume that $\nu \in S \implies \mathrm{cf}(\nu) \neq \lambda$. Contrary to the situation above, we show that after forcing with $j(D_S)$, a D_S-generic is added as well.
Fix a stationary subset $S \subset \kappa$ s.t $\nu \in S \implies \mathrm{cf}(\nu) \neq \lambda$ and note $D = D_S$, defined as above. We show that there is a complete projection $a\colon j(D) \longrightarrow D$.
For $q \in j(D)$ we define $\tilde{a}(q) \subset \kappa$ by

$$\alpha \in \tilde{a}(q) \iff j(\alpha) \in q.$$

For any $q \in j(D)$, $\tilde{a}(q)$ is a bounded subset of κ. Define $a(q) = \overline{\tilde{a}(q)}$, then $a(q)$ is a closed bounded subset of κ.
To see that $a\colon j(D) \longrightarrow D$ is well defined , i.e. that for any $q \in j(D)$, $a(q) \in D$, we need to show that $a(q) \cap S = \emptyset$.
Take $\alpha \in S$, we show that $\alpha \notin a(q)$. Clearly

$$\alpha \in S \implies j(\alpha) \in j(S) \overset{q\in j(D)}{\implies} j(\alpha) \notin q \implies \alpha \notin \tilde{a}(q).$$

Suppose by contradiction that $\alpha \in a(q)$. Then α is a limit point of $\tilde{a}(q)$. Note that since $\alpha \in S$, we have $\mu \equiv \mathrm{cf}(\alpha) \neq \lambda$. Hence we can write $\alpha = \sup_{\eta<\mu} \alpha_\eta$ where $\alpha_\eta \in \tilde{a}(q)$. i.e. $j(\alpha_\eta) \in q$. Now, as $\mu \neq \lambda$ is regular we have

$$\sup j\left(\left\langle \alpha_\eta; \eta < \mu \right\rangle\right) = \sup \left\langle j(\alpha_\eta); \eta < \mu \right\rangle,$$

therefore $j(\alpha) = \sup_{\eta<\mu} j(\alpha_\eta)$. This implies $j(\alpha) \in q$, as q is closed. Hence $\alpha \in \tilde{a}(q)$, in contradiction.
Now the rest of the proof (i.e. that a is a complete projection) follows just as in theorem 4.1.

5. Appendix

5.1. **Necessity of λ^+-distributivity.** In the preliminaries, section 0.3, we have seen that if $\mathscr{U}$ is an ultrafilter over λ, then the ultrapower embedding can be extended to an ultrapower embedding after a λ^+-distributive forcing extension. We show here that for the conclusions in 0.3, it is not enough to merely assume that $\mathscr{U}$ remains an ultrafilter in the generic extension (i.e. that no new subsets of λ are added). For a counter example, we use Magidor / Radin forcing for changing the cofinality of a regular cardinal. See [3] for Magidor and Radin forcing.
Let $\kappa > \lambda$ be an inaccessible cardinal and $u = \langle u_\alpha; \alpha < \lambda \rangle$ a measure sequence of length λ. Let $R = R_u$ be the corresponding Radin poset. Take a condition $p = \langle (u, A) \rangle$ where $A \subset V_\kappa$, $A \in \mathscr{F}(u)$ and $A \cap V_{\lambda+1} = \emptyset$. Then forcing with $R \downarrow p$ adds no new subsets to λ. In particular, if G is R-generic over V s.t $p \in G$, then

$\mathscr{U}$ is an ultrafilter over λ in $V[G]$ (though R is clearly not λ^+-distributive).
Consider the ultrapower embedding of $V[G]$,

$$\tilde{j}: V[G] \longrightarrow M \simeq \mathrm{Ult}(V[G], \mathscr{U}).$$

By elementarity, $M = M_0\left[\tilde{j}(G)\right]$ for some $M_0 \subset M$, where $M_0 = \hat{V}^M$ and $\tilde{j}(G)$ is $\tilde{j}(R)$-generic over M_0. Note that $\kappa > \lambda$ is inaccessible, hence $j(\kappa) = \kappa$. However, in $V[G]$, $\kappa > \lambda$ is a strong limit cardinal with cofinality λ, thus $\tilde{j}(\kappa) > \kappa$. Therefore $\tilde{j}$ cannot extend j.
More explicitly, if $\langle \kappa_\alpha; \alpha < \lambda \rangle$ is the cofinal λ-sequence added to κ, and we define

$$f\colon \lambda \longrightarrow V[G], \quad f(\alpha) = \kappa_\alpha.$$

For all $\alpha < \lambda$, $f(\alpha) \in V$, therefore $[f]_{\mathscr{U}}^{V[G]} \in \hat{V}^M = M_0$. However, $[f]_{\mathscr{U}}^{V[G]}$ is not equal to $[\check{g}]_{\mathscr{U}}^{V[G]}$ for any $g: \lambda \longrightarrow V$, $g \in V$.

5.2. **Swapping ultrapowers.** Suppose we have two measurable cardinals $\lambda < \kappa$. $\mathscr{U}$, $\mathscr{V}$ normal ultrafilter over λ, κ respectively, $j_{\mathscr{U}}: V \longrightarrow N$, $j_{\mathscr{V}}: V \longrightarrow M$ the corresponding ultrapower embeddings. Then we can further take ultrapowers $\mathrm{Ult}(N, j_{\mathscr{U}}(\mathscr{V}))$ and $\mathrm{Ult}(M, \mathscr{U})$ (where $\mathscr{U} = j_{\mathscr{V}}(\mathscr{U})$).
We claim that the two have the same transitive collapse and the resulting embeddings, $j_{\mathscr{U}}(j_{\mathscr{V}}) \circ j_{\mathscr{U}}$ and $j_{\mathscr{V}}(j_{\mathscr{U}}) \circ j_{\mathscr{V}}$, are the same. Furthermore, $j_{\mathscr{U}}(j_{\mathscr{V}})$, $j_{\mathscr{V}}(j_{\mathscr{U}})$ are essentially the same as $j_{\mathscr{V}}, j_{\mathscr{U}}$ respectively. That is, $j_{\mathscr{U}}(j_{\mathscr{V}}) = j_{\mathscr{V}}|_N$ and $j_{\mathscr{V}}(j_{\mathscr{U}}) = j_{\mathscr{U}}|_M$.
The argument clearly fails when $\mathscr{U}$, $\mathscr{V}$ are ultrafilters over the same cardinal λ. The only thing we use is that $\mathscr{V}$ is λ^+-complete. Hence the argument holds when $\mathscr{U}$ is an ultrafilter over λ and $\mathscr{V}$ a κ-complete ultrafilter over $P_\kappa\theta$ for some $\theta \geq \kappa$.
Let $j_\lambda: V \longrightarrow N \simeq \mathrm{Ult}(V, \mathscr{U})$, $j_\kappa: V \longrightarrow M \simeq \mathrm{Ult}(V, \mathscr{V})$, $j_\kappa(j_\lambda): M \longrightarrow N^M \simeq \mathrm{Ult}(M, \mathscr{U})$, $j_\lambda(j_\kappa): N \longrightarrow M^N \simeq \mathrm{Ult}(N, j_\lambda(\mathscr{V}))$.
We give a proof for the fact that $j_\lambda(j_\kappa) = j_\kappa|_N$ (which we use in section 3). The other arguments which we omit are similar, technical and categorically natural.
Note that $j_\kappa(\mathscr{U}) = \mathscr{U}$ and $j_\lambda(\kappa) = \kappa$ as $\kappa > \lambda$ is inaccessible.

Claim 5.1. $j_\lambda(\mathscr{V}) \subset \mathscr{V}$.

Proof. First, note that $\mathscr{V}$ concentrates on $I = \{\mu < \kappa; \mu \text{ is inaccessible}\}$, and for $X \subset I$ we have $X \subset j_\lambda(X)$. Thus

$$\forall X (X \in \mathscr{V} \implies j(X) \in \mathscr{V}).$$

Suppose now that $A \in j_\lambda(\mathscr{V})$, then $A = [f]_{\mathscr{U}}$ for some $f\colon \lambda \longrightarrow \mathscr{V}$. By λ^+-completeness of $\mathscr{V}$, we have that $B = \bigcap_{\eta<\lambda} f(\eta) \in \mathscr{V}$. Thus $j(B) \in \mathscr{V}$ and $j(B) \subset A$, hence $A \in \mathscr{V}$. ☐

Consider the map $M^N \longrightarrow M$ defined by $[f]_{j_\lambda(\mathscr{V})} \mapsto [f]_{\mathscr{V}}$. We show this is a $\in$-isomorphism from M^N to $\{[f]_{\mathscr{V}}; f\colon \lambda \longrightarrow N\} \subset M$.
Suppose $[g]_{j_\lambda(\mathscr{V})} \in [f]_{j_\lambda(\mathscr{V})}$ where $f, g \in N$, $f, g: \kappa \longrightarrow N$. Note

$$A = \{\mu; g(\mu) \in f(\mu)\} \in j_\lambda(\mathscr{V}).$$

By claim 5.1, $A \in \mathscr{V}$, hence $[g]_{\mathscr{V}} \in [f]_{\mathscr{V}}$. This also shows that the map is well defined, i.e.

$$[f]_{j_\lambda(\mathscr{V})} = [g]_{j_\lambda(\mathscr{V})} \implies [f]_{\mathscr{V}} = [g]_{\mathscr{V}}.$$

Similarly, if $[g]_{j_\lambda(\mathscr{V})} \notin [f]_{j_\lambda(\mathscr{V})}$, then $[g]_{\mathscr{V}} \notin [f]_{\mathscr{V}}$.
Thus the map is an injective $\in$-homomorphism, and it remains to show it is surjective, i.e. that for any

$f \in V$, $f\colon \lambda \longrightarrow N$, there is some $f' \in N$ s.t $[f]_{\mathscr{V}} = [f']_{\mathscr{V}}$.
Take such f, $f\colon \lambda \longrightarrow N$, then for each μ there is $\tilde{f}(\mu) : \lambda \longrightarrow V$ s.t $f(\mu) = \left[\tilde{f}(\mu)\right]_{\mathscr{U}}$. Define $\tilde{f}'\colon \lambda \longrightarrow V$ by

$$\tilde{f}'(\eta)(\mu) = \tilde{f}(\mu)(\eta),$$

that is, for $\eta < \lambda$, $\tilde{f}'(\eta) : \kappa \longrightarrow V$.
Note $f' = \left[\tilde{f}'\right]_{\mathscr{U}} \in N$. Thus $f'\colon \kappa \longrightarrow N$ and for any $\mu < \kappa$

$$f'(j_\lambda(\mu)) = \left[\tilde{f}'\right]_{\mathscr{U}}(j_\lambda(\mu)) = \left[\tilde{f}(\mu)\right]_{\mathscr{U}} = f(\mu).$$

As $\{\mu < \kappa;\, j_\lambda(\mu) = \mu\} \in \mathscr{V}$, we have $[f']_{\mathscr{V}} = [f]_{\mathscr{V}}$, as required.
Thus after the transitive collapses we get that M^N is contained in M. Also, for $x \in N$, if we consider the constant function $h \in N$, $h : \kappa \longrightarrow N$, $h(\mu) = x$. Then

$$j_\lambda(j_\kappa)(x) = [h]_{j_\lambda(\mathscr{V})} = [h]_{\mathscr{V}} = j_\kappa(x).$$

5.3. **Projection of a poset.** Recall the definitions of strategic closure and complete projections from the preliminaries.

Lemma 5.2. *Suppose P, Q are forcing notions and there is a complete projection $a\colon P \longrightarrow Q$. Then*

$$P \text{ is } \kappa\text{-strategically closed} \implies Q \text{ is } \kappa\text{-strategically closed}.$$

Proof. The idea is simple: player Even constructs along the play $\langle q_\alpha \rangle$ in $G_\kappa(Q)$ a corresponding play $\langle p_\alpha \rangle$ in $G_\kappa(P)$ such that $a(p_\alpha) \leq q_\alpha$. Then at each stage player Even uses the winning strategy in $G_\kappa(P)$ to make a move $p \in P$, and chooses $a(p)$ as the move in $G_\kappa(Q)$.
Fix a winning strategy S for player Even in $G_\kappa(P)$.
Consider the following strategy for player Even in $G_\kappa(Q)$, defined inductively on the stage β of the game.
Suppose $\beta < \kappa$ is even and we have $\langle q_\alpha;\, \alpha < \beta \rangle$, a descending chain of conditions in Q (which represents stage β of the game).
Suppose also (inductive hypothesis) we have a play in the game $G_\kappa(P)$, $\langle p_\alpha;\, \alpha < \beta \rangle$, in which player Even plays by the strategy S, and $\forall \alpha < \beta\, (a(p_\alpha) \leq q_\alpha)$.
Playing by the strategy S, we get a $p_\beta \in P$ such that $\forall \alpha < \beta \left(p_\beta \leq p_\alpha\right)$. Define $q_\beta = a\left(p_\beta\right)$ to be the move at stage β played by player Even in $G_\kappa(Q)$. The move is legit since

$$\forall \alpha < \beta \left(p_\beta \leq p_\alpha\right) \implies \forall \alpha < \beta \left(q_\beta \leq q_\alpha\right).$$

Let $q_{\beta+1} \in Q$ be the move played by player Odd, $q_{\beta+1} \leq q_\beta = a\left(p_\beta\right)$.
By the density condition for a, there is some $p_{\beta+1} \in P$ s.t $a\left(p_{\beta+1}\right) \leq q_{\beta+1} \wedge p_{\beta+1} \leq p_\beta$.
Thus $p_{\beta+1}$ is a legit move for player Odd in the corresponding game $G_\kappa(P)$. So we get two sequences $\langle q_\alpha : \alpha < \beta + 2 \rangle$, $\langle p_\alpha;\, \alpha < \beta + 2 \rangle$ as required by the inductive hypothesis for stage $\beta + 2$.
It is now simple to check that if player Even defines the sequence $\langle p_\alpha \rangle$ throughout the game as described above, at each stage the inductive hypothesis will be satisfied. Therefore this defines a winning strategy for player Even in the game $G_\kappa(Q)$. □

The same proof works when we have a complete embedding $i\colon Q \longrightarrow P$. The lemma above is the most common method for proving the strategic closure of forcing notions, where P is typically the Cohen poset. For example, for the poset S that adds a $\square(\kappa)$-sequence (see section 2), we show there is a

complete embedding of S into a κ-closed poset (which is of size κ, hence forcing isomorphic to κ-Cohen); for the poset $j(P)$, where P is Cohen forcing at κ, we show there is a complete projection of P^{λ} onto $j(P)$ (see section 1), where P^{λ} is forcing isomorphic to P.
We show below that the converse is also true, that is, a strategically closed poset can be embedded into Cohen forcing. So this method is in fact the only way to show a poset is strategically closed.

Lemma 5.3. *Suppose Q is a κ-strategically closed forcing notion of size κ. Then there is a complete projection $a\colon P \longrightarrow Q$, where $P = \mathrm{Cohen}(\kappa)$.*

Proof. We consider here P as

$$\{p\colon \mathrm{dom}p \longrightarrow \kappa; \mathrm{dom}p \in \kappa\},$$

and we note $P = \bigcup_{\alpha<\kappa} C_\alpha$ where

$$C_\alpha = \{p\colon \alpha \longrightarrow \kappa\}.$$

Fix a winning strategy S for player Even in the game $G_\kappa(Q)$. We inductively define $\left\{G_p; p\in P\right\}$, $G_p = \left\langle q^p_\eta; \eta \le 2\cdot \mathrm{dom}p\right\rangle$ s.t the following conditions hold:

(1) Each G_p is a play in the game $G_\kappa(Q)$ where player Even plays according to S.
(2) If $p \subset q$ then $G_p = G_q \restriction_{2\cdot \mathrm{dom}p}$.

Define $G_\emptyset = \langle 1_P\rangle$. Assume that for any $\alpha < \beta$ and for any $p \in C_\alpha$ we have constructed G_p. We wish to define G_p for each $p \in C_\beta$.
Suppose β is a limit ordinal. For any $p \in C_\beta$, consider the play $G = \bigcup_{\alpha<\beta} G_{p|\alpha}$. By conditions (1) and (2) above, G is a well defined play of length $2\cdot\beta = \beta$, where at each even stage player Even plays according to the strategy S. Thus the strategy S produces some q_β s.t $G_p = G \frown \left\langle q_\beta\right\rangle$ is a play.
Suppose β is a successor ordinal, $\beta = \alpha+1$ for $\alpha<\kappa$. We fix some $p \in C_\alpha$, and define $G_{p\frown\langle\xi\rangle}$ for $\xi<\kappa$. Let $G_p = \left\langle q^p_\eta; \eta \le 2\cdot\alpha\right\rangle$, and fix an enumeration of $Q\downarrow q^p_{2\cdot\alpha}$,

$$Q\downarrow q^p_{2\cdot\alpha} = \left\langle r^{q^p_{2\cdot\alpha}}_\xi; \xi<\kappa\right\rangle.$$

Now fix $\xi<\kappa$, and note $p' = p\frown\langle\xi\rangle$. Define $q^{p'}_{2\cdot\alpha+1} = r^{q^p_{2\cdot\alpha}}_\xi$ and $\forall\eta \le 2\cdot\alpha \left(q^{p'}_\eta = q^p_\eta\right)$. Then $\left\langle q^{p'}_\eta; \eta\le 2\cdot\alpha+1\right\rangle$ is a play in $G_\kappa(Q)$ where at each stage player Even plays according to S. Hence S produces some $q^{p'}_{2\cdot(\alpha+1)}$ s.t $G_{p'} = \left\langle q^{p'}_\eta; \eta \le 2\cdot(\alpha+1)\right\rangle$ is a play as required.
Now define a map $a\colon P\longrightarrow Q$ as follows. For $p\in P$, $\mathrm{dom}p = \alpha$, $G_p = \left\langle q^p_\eta; \eta\le 2\cdot\alpha\right\rangle$, define

$$a(p) = q^p_{2\cdot\alpha}.$$

By construction, $p \le q \implies a(p) \le a(q)$. Furthermore, for any $p\in P$, if $r\in Q$ and $r \le a(p)$, then $r = r^{a(p)}_\xi$ for some $\xi<\kappa$, hence $a(p\frown\langle\xi\rangle) \le r$. Thus the density condition of a is satisfied.

□

For each κ-strategically closed poset Q (of size κ), we thus have a projection $a\colon P\longrightarrow Q$. So there is a corresponding quotient poset $R = P/Q$ (more accurately, a Q-name for a poset). Recall some examples. If Q is the forcing for adding a $\square(\kappa)$-sequence, then R is the forcing for threading the sequence. If Q adds a non reflecting stationary subset $S\subset\kappa$, then R adds a club disjoint to S. If $Q = j(P)$ adds an unresolvable argument over κ, then R resolves the argument. In all these examples, we have seen that the combinatorial object added by Q cannot be destroyed by a sufficiently closed forcing notion (yet the proof was always trivially applicable to strategic closed forcings as well). From lemma 5.3 above we can generally deduce: if an object is not destroyed by closed forcings, it is not destroyed by strategically closed forcings as well.

Similarly, for cardinals $\kappa < \theta$, let $\mathrm{Col}(\kappa, \theta)$ be the Levi collapse. Then we have:

Proposition 5.4. *Let Q be a poset. Q is κ-strategically closed if and only if $\mathrm{Col}(\kappa, |Q|)$ completely projects onto Q.*

One direction follows from lemma 5.2. The other direction can be proved directly, analogously to lemma 5.3, but also follows from the following two observations together with lemma 5.3.

- For posets P, Q, there is a complete projection $a \colon P \longrightarrow Q$ if and only if there is a P-name $\dot{H}$ s.t $\Vdash_P \dot{H}$ is Q-generic over V and the following density condition holds: $\forall q \in Q \exists p \in P \left(p \Vdash q \in \dot{H}\right)$.
- For cardinals $\kappa < \theta$, $\mathrm{Col}(\kappa, \theta)$ is forcing isomorphic to $\mathrm{Col}(\kappa, \theta) * \mathrm{Cohen}(\kappa)$.

After forcing with $\mathrm{Col}(\kappa, |Q|)$, we have $|Q| = \kappa$ and Q is κ-strategically closed. Thus by lemma 5.3, there is a complete projection from $\mathrm{Cohen}(\kappa)$ onto Q. Hence by taking a further $\mathrm{Cohen}(\kappa)$-generic extension, we get a Q-generic. Thus any $\mathrm{Col}(\kappa, |Q|)$-generic extension adds a Q-generic, and the density condition holds, as in the intermediate extension we had a complete projection $\mathrm{Cohen}(\kappa) \longrightarrow Q$.

5.4. **Preservation of stationary sets.**

Lemma 5.5. *Suppose κ is inaccessible, $\lambda < \kappa$, $S \subset \kappa$ a stationary subset with $\mathrm{cof}(S) = \lambda$ and P is a λ^+-closed forcing notion. Then $S \subset \kappa$ remains stationary after forcing with P.*

Proof. Let $\dot{C}$ be a P-name s.t $\Vdash \dot{C} \subset \kappa$ is a club. We must show that $\Vdash \dot{C} \cap S \neq \emptyset$.
Since κ is inaccessible, the set
$$E = \left\{\eta < \kappa; \left\langle V_\eta, P, \dot{C}, S, \eta \right\rangle \prec \left\langle V_\kappa, P, \dot{C}, S, \kappa \right\rangle \right\}$$
is a club subset of κ. Since S is stationary, we can pick $\eta \in S \cap E$.
$\eta \in S \implies \mathrm{cf}(\eta) = \lambda$. Let $\langle \eta_\alpha; \alpha < \lambda \rangle$ be a cofinal sequence in η.
$P^{V_\eta} \Vdash \dot{C}^{V_\eta}$ is a club in η. We can construct a descending sequence of conditions $\langle p_\alpha; \alpha < \lambda \rangle \subset P^{V_\eta}$ and ordinals $\langle \mu_\alpha; \alpha < \lambda \rangle$, s.t $p_\alpha \Vdash \mu_\alpha \in \dot{C}^{V_\eta}$. The construction is possible since for any $\delta < \lambda$, $\langle p_\alpha; \alpha < \delta \rangle \in V_\eta$, and at limit stages we use the λ-closure of P^{V_η}. Note that $P^{V_\eta} = P \cap V_\eta$ and if $p_\alpha \Vdash_{P^{V_\eta}} \mu_\alpha \in \dot{C}^{V_\eta}$ then $p_\alpha \Vdash_P \mu_\alpha \in \dot{C}$.
By λ^+-closure of P, we have $p \in P$ s.t $\forall \alpha < \lambda \, (p \leq p_\alpha)$. Since $\dot{C}$ is forced to be closed, we have $p \Vdash \eta \in \dot{C}$, hence $p \Vdash \eta \in \dot{C} \cap S$.
The construction above in fact shows that such a p can be found extending any given condition $q \in P$, hence $P \Vdash \dot{C} \cap S \neq \emptyset$. □

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