ON THE PROOF THAT A TREE WITH AN ASCENT PATH IS NOT SPECIAL

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Abstract. We give a proof that a tree with an ascent path is not special (Shelah-Stanley [3]) using ultrapowers.

Definition 1. *(Devlin* [1]*, Shelah-Stanley* [3]*) Let $\lambda < \kappa$ be regular cardinals. Suppose T is a tree of height κ. A λ-ascent path through T is a sequence $x = \left\langle x_\xi^\alpha;\, \alpha < \kappa,\, \xi < \lambda \right\rangle$ satisfying:*

- $(\forall \alpha < \kappa)\,(\forall \xi < \lambda)\left(x_\xi^\alpha \in T_\alpha\right)$;
- $(\forall \alpha < \beta < \kappa)\,(\exists \delta < \lambda)\,(\forall \xi > \delta)\ \left(x_\xi^\alpha <_T x_\xi^\beta\right)$.

Theorem 1. *(Lemma 3 of* [3]*) If T is a κ^+-Aronszajn tree with a λ-ascent path, and $\lambda \neq \mathrm{cf}\kappa$, then T is not special.*

The proof provided in [3] (page 6) is fairly short and simple, yet somewhat tricky (in the paper it is followed by a discussion of the development of the proof through some false attempts). The proof provided below is essentially the same, but using an ultrapower to absorb most of the combinatorics. The arguments below rely on a single fact, that if $\mathcal{U}$ is an ultrafilter over a regular cardinal λ, κ is a cardinal with cofinality different than λ, and $j\colon V \to \mathrm{Ult}(V,\mathcal{U})$ is the ultrapower embedding, then $j''\kappa$ is cofinal in $j(\kappa)$. This is precisely when the assumption $\lambda \neq \mathrm{cf}\kappa$ of theorem 1 is invoked. (This assumption is necessary, as there can be special κ^+-Aronszajn trees with a κ-ascent path.

Remark 1. *If T has a λ-ascent path, and $\mathcal{U}$ is a non trivial ultrafilter over λ, then the tree $\mathrm{Ult}(T,\mathcal{U})$ has a branch. Note that the ultrapower is not assumed to be well founded, and the branch is not in $\mathrm{Ult}(V,\mathcal{U})$. See* [2] *for more on this approach.*

By the remark, the following proposition proves theorem 1:

Proposition 1. *Suppose λ is a regular cardinal, $\lambda < \kappa$ and $\mathrm{cf}\kappa \neq \lambda$. Suppose T is a special κ^+-tree and $\mathcal{U}$ is an ultrafilter over λ. Then $\mathrm{Ult}(T,\mathcal{U})$ has no branch.*

Proof. Let $f\colon T \to \kappa$ be a specializing function. Consider the embedding $j\colon V \to \mathrm{Ult}\,(V,\mathcal{U})$. Assume for contradiction that there is a cofinal branch b in the tree $j\,(T)$. In particular, for any $\alpha < \kappa^+$, we have $b_{j(\alpha)} \in j\,(T_\alpha)$, and $j\,(f)\,\left(b_{j(\alpha)}\right) \in j\,(\kappa)$. Since $\mathrm{cf}\,(\kappa) \neq \lambda$,

$$j''\kappa \text{ is cofinal in } j\,(\kappa)\,.$$

So for any $\alpha < \kappa^+$, there is $\theta < \kappa$ such that $j(f)(b_{j(\alpha)}) < j(\theta)$. This gives a map $\kappa^+ \to \kappa$ sending α to such θ. Fix some $\theta < \kappa$ and a cofinal $X \subset \kappa^+$ such that

$$\forall \alpha \in X\left(j\,(f)\,\left(b_{j(\alpha)}\right) < j\,(\theta)\right).$$

Take $\alpha \in \lim X$ with $\mathrm{cf}\alpha > \lambda, |\theta|$.

Since $\alpha \in \lim X$, there are cofinally many $\xi < \alpha$ with $j(f)\left(b_{j(\xi)}\right) < j(\theta)$. Since $\mathrm{cf}\alpha > \lambda$, then $j''\alpha$ is cofinal in $j(\alpha)$. So there are cofinally many $\xi < j(\alpha)$ s.t. $j(f)(b_\xi) < j(\theta)$. This statement can be written as follows:

$$\text{There are cofinally many } \xi < j(\alpha) \text{ s.t. } j(f)(\mathrm{Pr}_\xi(b_{j(\alpha)})) < j(\theta).$$

Where Pr_ξ is the projection function to level ξ (for the tree $j(T)$). Note that the latter statement is expressible in $\mathrm{Ult}(V,\mathcal{U})$. By elementarity, $j(f)$ is injective on $\{\mathrm{Pr}_\xi(b_{j(\alpha)});\, \xi < j(\alpha)\}$. However, by the choice of α, $\mathrm{Ult}(V,\mathcal{U}) \models \mathrm{cf}(j(\alpha)) > |j(\theta)|$. In contradiction. □

Remark 2. *Theorem 1 can be generalized to limit cardinals. The same arguments as above will give the following:*

Assume $\lambda < \kappa$ are regular limit cardinals, and $\mathcal{U}$ is an ultrafilter over λ. Let S be a stationary subset of κ such that no point in S has cofinality λ. Assume there is a partial regressive specializing function $f\colon T \to \kappa$ defined on the levels in S. That is, for any $\alpha \in S$ and $x \in T_\alpha$, $f(x) < \alpha$ is defined, and $x \lhd y \implies f(x) \neq f(y)$. (In other words f is a weakly specializing function on S?)
Then $\mathrm{Ult}(T,\mathcal{U})$ *has no branch.*

References

[1] Devlin, K. J.: Reduced products of $\aleph_2$-trees, Fundamenta Mathematicae 118 (1983), pp 129-134.
[2] Shani, A. Arch. Math. Logic vol. 55 (2016), pp. 835-845.
[3] Shelah, S., Stanley, L.: Weakly compact cardinals and nonspecial Aronszajn trees. Proc. Amer. Math. Soc., 104(3):887-897, 1988.

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