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ON THE COMPLEXITY OF THE CLASSIFICATION PROBLEM FOR TORSION-FREE ABELIAN GROUPS OF FINITE RANK

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§1. Introduction. In this paper, we shall discuss some recent contributions to the project [15, 14, 2, 18, 22, 23] of explaining why no satisfactory system of complete invariants has yet been found for the torsion-free abelian groups of finite rank $n \geq 2$. Recall that, up to isomorphism, the torsion-free abelian groups of rank n are exactly the additive subgroups of the n-dimensional vector space $\mathbb{Q}^n$ which contain n linearly independent elements. Thus the collection of torsion-free abelian groups of rank at most n can be naturally identified with the set $S(\mathbb{Q}^n)$ of all nontrivial additive subgroups of $\mathbb{Q}^n$. In 1937, Baer [4] solved the classification problem for the class $S(\mathbb{Q})$ of rank 1 groups as follows.

Let $\mathbb{P}$ be the set of primes. If G is a torsion-free abelian group and $0 \neq x \in G$, then the *p-height* of x is defined to be

$$h_x(p) = \sup\{n \in \mathbb{N} \mid \text{There exists } y \in G \text{ such that } p^n y = x\ \} \in \mathbb{N} \cup \{\infty\};$$

and the *characteristic* $\chi(x)$ of x is defined to be the function

$$\langle h_x(p) \mid p \in \mathbb{P} \rangle \in (\mathbb{N} \cup \{\infty\})^{\mathbb{P}}.$$

Two functions $\chi_1, \chi_2 \in (\mathbb{N} \cup \{\infty\})^{\mathbb{P}}$ are said to be *similar* or to *belong to the same type*, written $\chi_1 \equiv \chi_2$, iff

(a) $\chi_1(p) = \chi_2(p)$ for almost all primes p; and
(b) if $\chi_1(p) \neq \chi_2(p)$, then both $\chi_1(p)$ and $\chi_2(p)$ are finite.

Clearly $\equiv$ is an equivalence relation on $(\mathbb{N} \cup \{\infty\})^{\mathbb{P}}$. If G is a torsion-free abelian group and $0 \neq x \in G$, then the *type* $\tau(x)$ of x is defined to be the $\equiv$-equivalence class containing the characteristic $\chi(x)$.

Now suppose that $G \in S(\mathbb{Q})$ is a rank 1 group. Then it is easily checked that $\tau(x) = \tau(y)$ for all $0 \neq x, y \in G$. Hence we can define the *type* $\tau(G)$ of G to be $\tau(x)$, where x is any non-zero element of G. In [4], Baer proved that $\tau(G)$ is a complete invariant for the isomorphism problem for $S(\mathbb{Q})$.

THEOREM 1.1 (Baer [4]). *If $G, H \in S(\mathbb{Q})$, then $G \cong H$ iff $\tau(G) = \tau(H)$.*

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However, the situation is much less satisfactory in the case of the torsion-free abelian groups of rank $n \geq 2$. In the late 1930s, Kurosh [20] and Malcev [21] found complete invariants for these groups consisting of equivalence classes of infinite sequences $\langle M_p \mid p \in \mathbb{P} \rangle$ of matrices, where each $M_p \in GL_n(\mathbb{Q}_p)$. However, as Fuchs [11, Section 93] remarks, the associated equivalence relation is so complicated that the problem of deciding whether two sequences are equivalent is as difficult as that of deciding whether the corresponding groups are isomorphic. It is natural to ask whether the classification problem for $S(\mathbb{Q}^n)$ is genuinely more difficult when $n \geq 2$. Of course, if we wish to show that the classification problem for the groups of rank $n \geq 2$ is intractible, it is not enough merely to prove that there are 2^ω such groups up to isomorphism. For there are 2^ω pairwise nonisomorphic groups of rank 1 and we have already pointed out that Baer has given a satisfactory classification for this class of groups. In this paper, following Friedman-Stanley [10] and Hjorth-Kechris [15], we shall use the more sensitive notions of descriptive set theory to measure the complexity of the classification problem for the groups of rank $n \geq 2$.

Let X be a standard Borel space; i.e., a Polish space equipped with its associated σ-algebra of Borel subsets. Then a *Borel equivalence relation* on X is an equivalence relation $E \subseteq X^2$ which is a Borel subset of X^2. If E, F are Borel equivalence relations on the standard Borel spaces X, Y respectively, then we say that E is *Borel reducible* to F and write $E \leq_B F$ if there exists a Borel function $f: X \to Y$ such that xEy iff $f(x)Ff(y)$. We say that E and F are *Borel bireducible* and write $E \sim_B F$ if both $E \leq_B F$ and $F \leq_B E$. Finally we write $E <_B F$ if both $E \leq_B F$ and $F \not\leq_B E$. Most of the Borel equivalence relations that we shall consider in this paper arise from group actions as follows. Let G be a countable group equipped with the discrete topology. Then a *standard Borel G-space* is a standard Borel space X equipped with a Borel action $(g, x) \mapsto g.x$ of G on X. The corresponding G-orbit equivalence relation on X, which we shall denote by E_G^X, is a *countable Borel equivalence relation*; i.e., a Borel equivalence relation E such that every E-equivalence class is countable. Conversely, by Feldman-Moore [9], if E is an arbitrary countable Borel equivalence relation on the standard Borel space X, then there exists a countable group G and a Borel action of G on X such that $E = E_G^X$.

To see how the classification problem for torsion-free abelian groups fits into this context, note that $S(\mathbb{Q}^n)$ is a Borel subset of the Polish space $\mathcal{P}(\mathbb{Q}^n)$ of all subsets of $\mathbb{Q}^n$ and so $S(\mathbb{Q}^n)$ is a standard Borel space. (Here we are identifying $\mathcal{P}(\mathbb{Q}^n)$ with the space $2^{\mathbb{Q}^n}$ of all functions $h: \mathbb{Q}^n \to \{0, 1\}$ equipped with the product topology.) Furthermore, $S(\mathbb{Q}^n)$ is a standard Borel $GL_n(\mathbb{Q})$-space under the action induced from the natural action of $GL_n(\mathbb{Q})$ on the vector space $\mathbb{Q}^n$; and it is easily checked that if $A, B \in S(\mathbb{Q}^n)$, then $A \cong B$ iff there exists an element $\varphi \in GL_n(\mathbb{Q})$ such that $\varphi(A) = B$.

Thus the isomorphism relation on $S(\mathbb{Q}^n)$ is a countable Borel equivalence relation.

Notation 1.2. $\cong_n$ denotes the isomorphism relation on $S(\mathbb{Q}^n)$.

It is clear that $(\cong_n) \leq_B (\cong_{n+1})$ for all $n \geq 1$; and our earlier question on the difficulty of the classification problem for the torsion-free abelian groups of rank $n \geq 2$ can be rephrased as the question of whether $(\cong_1) <_B (\cong_n)$ when $n \geq 2$. Before we discuss Hjorth's solution [14] of this problem and some more recent developments in this area, it will be helpful to first give a brief account of some of the general theory of countable Borel equivalence relations. (A detailed development of the theory can be found in Jackson-Kechris-Louveau [16].)

The least complex countable Borel equivalence relations are those which are *smooth*; i.e., those countable Borel equivalence relations E on a standard Borel space X for which there exists a Borel function $f : X \to Y$ into a standard Borel space Y such that xEy iff $f(x) = f(y)$. Next in complexity come those countable Borel equivalence relations E such that E is Borel bireducible with the *Vitali equivalence relation* E_0 defined on $2^{\mathbb{N}}$ by xE_0y iff $x(n) = y(n)$ for almost all n. More precisely, by Harrington-Kechris-Louveau [13], if E is a countable Borel equivalence relation, then E is non-smooth iff $E_0 \leq_B E$. Furthermore, by Dougherty-Jackson-Kechris [8], if E is a countable Borel equivalence relation on a standard Borel space X, then the following three properties are equivalent:

(1) $E \leq_B E_0$.
(2) E is *hyperfinite*; i.e., there exists an increasing sequence
$$F_0 \subseteq F_1 \subseteq \cdots \subseteq F_n \subseteq \dots$$
of finite Borel equivalence relations on X such that $E = \bigcup_{n\in\mathbb{N}} F_n$. (Here an equivalence relation F is said to be *finite* iff every F-equivalence class is finite.)
(3) There exists a Borel action of $\mathbb{Z}$ on X such that $E = E^X_{\mathbb{Z}}$.

It turns out that there is also a most complex countable Borel equivalence relation E_∞, which is *universal* in the sense that $F \leq_B E_\infty$ for every countable Borel equivalence relation F, and that $E_0 <_B E_\infty$. (Clearly this universality property uniquely determines E_∞ up to Borel bireducibility.) E_∞ has a number of natural realisations in many areas of mathematics, including algebra, topology and recursion theory. (See Jackson-Kechris-Louveau [16].) For example, E_∞ is Borel bireducible to both the isomorphism relation for finitely generated groups [24] and the isomorphism relation for fields of finite transcendence degree [25].

For many years, it was an open problem whether, up to Borel bireducibility, there existed infinitely many countable Borel equivalence relations E such that $E_0 <_B E <_B E_\infty$. This problem was finally resolved by Adams-Kechris

[2], who used Zimmer's superrigidity theory [27] to show that there are actually 2^{ω} such relations E up to Borel bireducibility. Of these, the only ones which have been extensively studied are the treeable relations, which were originally introduced by Adams [1]. The countable Borel equivalence relation E on the standard Borel space X is said to be *treeable* iff there exists a Borel relation $R \subseteq X \times X$ such that:

(a) $\langle X; R\rangle$ is an acyclic graph; and
(b) the connected components of $\langle X; R\rangle$ are precisely the E-equivalence classes.

For example, whenever a countable free group Γ has a free Borel action on a standard Borel space X, then the corresponding orbit equivalence relation E_{Γ}^{X} is treeable. (Here Γ is said to act *freely* on X iff $g.x \neq x$ for all $1 \neq g \in \Gamma$ and $x \in X$.) The class of treeable relations is not nearly so well understood as that of the hyperfinite relations. For example, it is known that there exists a universal treeable countable Borel equivalence relation $E_{T\infty}$; but it remains open whether there exists a (necessarily treeable) countable Borel equivalence relation E such that

$$E_0 <_B E <_B E_{T\infty},$$

or even whether $E_{T\infty} \leq_B E$ for every non-hyperfinite countable Borel equivalence relation E. (Fuller accounts of the notion of treeability can be found in Hjorth-Kechris [15] and Jackson-Kechris-Louveau [16].)

Returning to our discussion of the complexity of the isomorphism relation $\cong_n$ on $S(\mathbb{Q}^n)$, it is easily checked that Baer's classification of the rank 1 groups implies that $(\cong_1) \sim_B E_0$. In [15], Hjorth-Kechris conjectured that $(\cong_n) \sim_B E_\infty$ for all $n \geq 2$. Of course, if true, this would have explained the failure to find a satisfactory system of complete invariants for the torsion-free abelian groups of rank $n \geq 2$, since nobody expects such a system to exist for the class of finitely generated groups. In [14], Hjorth provided some evidence for this conjecture by proving that if $n \geq 2$, then $\cong_n$ is not hyperfinite and hence $(\cong_1) <_B (\cong_n)$. (For $n \geq 3$, Hjorth proved the stronger result that $\cong_n$ is not treeable. More recently, Kechris [18] has shown that $\cong_2$ is also not treeable.) Later in [2], Adams-Kechris used Zimmer's superrigidity theory [27] to prove the intriguing result that

$$(\cong_1^*) <_B (\cong_2^*) <_B \cdots <_B (\cong_n^*) <_B \cdots$$

where $(\cong_n^*)$ is the restriction of the isomorphism relation to the Borel subset $S^*(\mathbb{Q}^n)$ consisting of the *rigid* groups $A \in S(\mathbb{Q}^n)$. (Here an abelian group A is said to be rigid if its only automorphism are the obvious ones: $a \mapsto a$ and $a \mapsto -a$.) In particular, none of the relations $\cong_n^*$ is a universal countable Borel equivalence relation. It was not clear whether the Adams-Kechris result should be regarded as evidence for or against the Hjorth-Kechris conjecture, since very little was known concerning the relationship between

$\cong_n^*$ and $\cong_m$ for $n, m \geq 1$. Of course, it is clear that $(\cong_n^*) \leq_B (\cong_n)$ for all $n \geq 1$; and it is easily seen that $(\cong_1^*) \sim_B E_0$ and so $(\cong_1^*) \sim_B (\cong_1)$. But, apart from these easy observations, essentially nothing else was known. Very recently, Thomas [22] disproved the Hjorth-Kechris conjecture by showing that $(\cong_3^*) \not\leq_B (\cong_2)$. However, Thomas [22] left open the possibility that there might exist an integer $n > 2$ such that $(\cong_n) \sim_B E_\infty$. This possibility was finally eliminated in Thomas [23].

THEOREM 1.3 ([23]). *$(\cong_{n+1}^*) \not\leq_B (\cong_n)$ for all $n \geq 1$.*

As an immediate consequence, we obtain that the classification problem for $S(\mathbb{Q}^{n+1})$ is strictly more complex than that for $S(\mathbb{Q}^n)$.

THEOREM 1.4 ([23]). *$(\cong_n) <_B (\cong_{n+1})$ for all $n \geq 1$.*

The rest of this paper is organised as follows. In Section 2, we shall sketch the proof of Hjorth's Theorem. In Section 3, we shall discuss the notion of a cocycle of a group action and sketch the proof of the Adams-Kechris Theorem. Finally in Section 4, we shall sketch the proof of Theorem 1.3.

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§2. Invariant probability measures and Hjorth's Theorem. In this section, we shall sketch the proof of Hjorth's Theorem that $(\cong_1) <_B (\cong_n)$ for all $n \geq 2$.

THEOREM 2.1 (Hjorth [14]). *If $n \geq 2$, then $\cong_n$ is not hyperfinite.*

Hjorth's result ultimately depends upon the fact that $\cong_n$ is the orbit equivalence relation of the action of $GL_n(\mathbb{Q})$ on $S(\mathbb{Q}^n)$ and that $GL_n(\mathbb{Q})$ is a nonamenable group for all $n \geq 2$. (Recall that a countable group G is *amenable* iff there exists a finitely additive G-invariant probability measure $\nu : \mathcal{P}(G) \to [0, 1]$ defined on every subset of G.) However, these observations alone are not enough to conclude that $\cong_n$ is not hyperfinite; since for every countable group G, there exists a standard Borel G-space X such that the relation E_G^X is smooth. For example, we can take $X = G \times \mathbb{R}$ equipped with the Borel action $g.(h, r) = (gh, r)$.

PROPOSITION 2.2 ([16]). *Let G be a countable nonamenable group and let X be a standard Borel G-space with a invariant probability measure ν. If G acts freely on X, then E_G^X is not hyperfinite.*

Here the probability measure ν on the standard Borel G-space X is said to be *invariant* iff $\nu(g(B)) = \nu(B)$ for every $g \in G$ and Borel subset $B \subseteq X$. The probability measure ν on X is said to be *nonatomic* if $\nu(\{x\}) = 0$ for all $x \in X$. Of course, if the probability measure ν on X is invariant under the free action of an infinite group, then ν is necessarily nonatomic.

Unfortunately we cannot apply Proposition 2.2 directly to the action of $GL_n(\mathbb{Q})$ on $S(\mathbb{Q}^n)$. Firstly, the action is not free. In fact, for each rank n

group $A \in S(\mathbb{Q}^n)$, the stabiliser of A in $GL_n(\mathbb{Q})$ is precisely the automorphism group $\mathrm{Aut}(A)$ of A. Secondly, Hjorth has shown that there does not exist a $GL_n(\mathbb{Q})$-invariant nonatomic probability measure on $S(\mathbb{Q}^n)$. However, the following result will enable us to work instead with the induced action of the subgroup $SL_n(\mathbb{Z})$ on a suitably chosen Borel subset of $S(\mathbb{Q}^n)$.

PROPOSITION 2.3 ([16]). *Let E be a hyperfinite equivalence relation on the standard Borel space X.*

(i) *If F is a Borel equivalence relation on X and $F \subseteq E$, then F is hyperfinite.*
(ii) *If B is a Borel subset of X, then the restriction $E \restriction B$ is hyperfinite.*

Fix some $n \geq 2$ and let $S(\mathbb{Q}^n, \mathbb{Z}^n)$ be the $SL_n(\mathbb{Z})$-invariant Borel subset of $S(\mathbb{Q}^n)$ consisting of those subgroups A such that $\mathbb{Z}^n \leqslant A$.

THEOREM 2.4 ([14]). *There exists an $SL_n(\mathbb{Z})$-invariant nonatomic probability measure ν on $S(\mathbb{Q}^n, \mathbb{Z}^n)$ such that*

$$\nu(\{A \in S(\mathbb{Q}^n, \mathbb{Z}^n) \mid A \textit{ is rigid}\}) = 1.$$

Before we sketch the proof of Theorem 2.4, we shall show how to derive Hjorth's Theorem.

PROOF OF THEOREM 2.1. Let $S^*(\mathbb{Q}^n, \mathbb{Z}^n)$ be the Borel subset consisting of the rigid groups $A \in S(\mathbb{Q}^n, \mathbb{Z}^n)$; and let F be the orbit equivalence relation of the action of $SL_n(\mathbb{Z})$ on $S^*(\mathbb{Q}^n, \mathbb{Z}^n)$. Since the stabiliser of each group $A \in S^*(\mathbb{Q}^n, \mathbb{Z}^n)$ in $SL_n(\mathbb{Z})$ is $\mathrm{Aut}(A) = \{\pm I\}$, it follows that F is the orbit equivalence relation arising from the induced free action of $PSL_n(\mathbb{Z}) = SL_n(\mathbb{Z})/\{\pm I\}$ on $S^*(\mathbb{Q}^n, \mathbb{Z}^n)$. As $PSL_n(\mathbb{Z})$ is nonamenable, Proposition 2.2 implies that F is not hyperfinite. Since $F \subseteq (\cong_n \restriction S^*(\mathbb{Q}^n, \mathbb{Z}^n))$, Proposition 2.3 implies that $\cong_n$ is also not hyperfinite. ⊣

Let $\mathcal{S}_n$ be the standard Polish space consisting of all subgroups of the quotient group $\mathbb{Q}^n/\mathbb{Z}^n$ and let $\pi \colon \mathbb{Q}^n \to \mathbb{Q}^n/\mathbb{Z}^n$ be the canonical surjection. Then we can define a Borel bijection $S(\mathbb{Q}^n, \mathbb{Z}^n) \to \mathcal{S}_n$ by $A \mapsto \pi(A)$. Let Γ be the dual group of $\mathbb{Q}^n/\mathbb{Z}^n$; i.e., the space of all homomorphisms $\psi \colon \mathbb{Q}^n/\mathbb{Z}^n \to \mathbb{R}/\mathbb{Z}$, equipped with the topology of pointwise convergence and the group operation of pointwise addition. Since $\mathbb{Q}^n/\mathbb{Z}^n$ is discrete, it follows that Γ is a compact group. Let m be the Haar measure on Γ. Let $k : \Gamma \to \mathcal{S}_n$ be the Borel map assigning the subgroup

$$\ker(\psi) = \{h \in \mathbb{Q}^n/\mathbb{Z}^n \mid \psi(h) = 0\}$$

to each $\psi \in \Gamma$; and let $\widehat{\nu}$ be the probability measure defined on $\mathcal{S}_n$ by

$$\widehat{\nu}(B) = m(k^{-1}(B)) = m(\{\psi \in \Gamma \mid \ker(\psi) \in B\})$$

for each Borel subset $B \subseteq \mathcal{S}_n$. Then ν is the corresponding Borel probability measure on $S(\mathbb{Q}^n, \mathbb{Z}^n)$ induced by the Borel bijection $A \mapsto \pi(A)$.

It is clear that ν is nonatomic. Using the uniqueness of the Haar measure m, it follows quite easily that ν is $SL_n(\mathbb{Z})$-invariant. (For the details, see

Hjorth [14, Lemma 4.2].) Finally the next lemma says that the collection of "obviously rigid" groups has ν-measure 1.

LEMMA 2.5 ([14]). *Let Z_n be the Borel subset consisting of those subgroups $A \in S(\mathbb{Q}^n, \mathbb{Z}^n)$ which satisfy the following conditions*:

(i) *$pA \neq A$ for all primes $p \in \mathbb{P}$.*

(ii) *If $a, b \in A$ are linearly independent, then $\tau(a) \neq \tau(b)$.*

Then $\nu(Z_n) = 1$.

To see that each $A \in Z_n$ is rigid, suppose that $\varphi \in \mathrm{Aut}(A)$. If $0 \neq a \in A$, then $\tau(\varphi(a)) = \tau(a)$ and so Condition 2.5(ii) implies that there exists $0 \neq \lambda_a \in \mathbb{Q}$ such that $\varphi(a) = \lambda_a.a$. It follows easily that there exists a fixed $\lambda \in \mathbb{Q}$ such that $\lambda_a = \lambda$ for all $0 \neq a \in A$. Since $pA \neq A$ for all primes $p \in \mathbb{P}$, it follows that $\lambda \in \{\pm 1\}$.

Now suppose that $a \in \mathbb{Q}^n \smallsetminus \mathbb{Z}^n$ and that $\pi(a)$ has order $n > 0$ in $\mathbb{Q}^n/\mathbb{Z}^n$. Let $\Gamma_a = \{\psi \in \Gamma \mid \psi(a) = 0\}$. Then

$$\nu(\{A \in S(\mathbb{Q}^n, \mathbb{Z}^n) \mid a \in A\}) = \widehat{\nu}(\Gamma_a) = [\Gamma \colon \Gamma_a] = 1/n.$$

Furthermore, if $a, b \in \mathbb{Q}^n \smallsetminus \mathbb{Z}^n$ are linearly independent, then the events

$$\{A \in S(\mathbb{Q}^n, \mathbb{Z}^n) \mid a \in A\} \quad \text{and} \quad \{A \in S(\mathbb{Q}^n, \mathbb{Z}^n) \mid b \in A\}$$

are ν-independent.

It follows easily that if $p \in \mathbb{P}$, then $pA \neq A$ for ν-almost all $A \in S(\mathbb{Q}^n, \mathbb{Z}^n)$. Now let $e_1, \dots, e_n$ be the standard basis of $\mathbb{Q}^n$; and for each $p \in \mathbb{P}$, let

$$E_p = \{A \in S(\mathbb{Q}^n, \mathbb{Z}^n) \mid e_1/p \in A \text{ and } e_2/p \notin A\}.$$

Then $\nu(E_p) = (p-1)/p^2$ and the events $\{E_p \mid p \in \mathbb{P}\}$ are ν-independent. Since

$$\sum_{p \in \mathbb{P}} (p-1)/p^2 = \infty,$$

the Second Borel-Cantelli Lemma [5, Theorem 4.4] implies that

$$\nu\left(\bigcap_{n=1}^{\infty} \bigcup_{p>n} E_p\right) = 1.$$

Hence $\tau(e_1) \neq \tau(e_2)$ in ν-almost all $A \in S(\mathbb{Q}^n, \mathbb{Z}^n)$. Since $SL_n(\mathbb{Z})$ acts transitively on the set of ordered pairs of 1-dimensional subspaces of $\mathbb{Q}^n$ and ν is $SL_n(\mathbb{Z})$-invariant, it follows easily that

$$\nu(\{A \in S(\mathbb{Q}^n, \mathbb{Z}^n) \mid \tau(a) \neq \tau(b) \text{ for all linearly independent } a, b \in A\}) = 1.$$

This completes the proof of Lemma 2.5.

In the remaining sections of this paper, we shall need to work with ergodic probability measures. The G-invariant probability measure μ on the standard Borel G-space X is said to be *ergodic* iff for every G-invariant Borel subset $B \subseteq X$, either $\mu(B) = 0$ or $\mu(B) = 1$. It is well-known that the following two properties are equivalent:

(i) μ is ergodic.
(ii) If Y is a standard Borel space and $f : X \to Y$ is a G-invariant Borel function, then there exists a G-invariant Borel subset $M \subseteq X$ with $\mu(M) = 1$ such that $f \restriction M$ is a constant function.

It is easily seen that Hjorth's measure ν on $S(\mathbb{Q}^n, \mathbb{Z}^n)$ is not ergodic. For example, let $p \in \mathbb{P}$ be a prime and let

$$B = \{A \in S(\mathbb{Q}^n, \mathbb{Z}^n) \mid \mathbb{Z}^n \leqslant pA\}.$$

Then B is an $SL_n(\mathbb{Z})$-invariant Borel subset of $S(\mathbb{Q}^n, \mathbb{Z}^n)$ such that $\nu(B) = (1/p)^n$. However, using the ergodic decomposition [26] of the ν-preserving action of $SL_n(\mathbb{Z})$ on $S^*(\mathbb{Q}^n, \mathbb{Z}^n)$, Adams-Kechris proved the following result.

THEOREM 2.6 ([2]). *There exists an $SL_n(\mathbb{Z})$-invariant Borel subset X_n of $S(\mathbb{Q}^n, \mathbb{Z}^n)$ with the following properties.*
(i) *Each $A \in X_n$ is rigid.*
(ii) *There exists an $SL_n(\mathbb{Z})$-invariant ergodic nonatomic probability measure μ on X_n.*

§**3. Cocycles of group actions and the Adams-Kechris theorem.** In this section, we shall sketch the proof of the Adams-Kechris theorem.

THEOREM 3.1 (Adams-Kechris [2]). $(\cong_n^*) <_B (\cong_{n+1}^*)$ *for all $n \geq 1$.*

Theorem 3.1 is an easy consequence of the following more technical result.

THEOREM 3.2 ([2]). *Let $m \geq 3$ and let X be a standard Borel $SL_m(\mathbb{Z})$-space with an invariant ergodic probability measure μ. Suppose that $1 \leq n < m$ and that $f : X \to S^*(\mathbb{Q}^n)$ is a Borel function such that $xE^X_{SL_m(\mathbb{Z})}y$ implies $f(x) \cong_n^* f(y)$. Then there exists an $SL_m(\mathbb{Z})$-invariant Borel subset M with $\mu(M) = 1$ such that f maps M into a single $\cong_n^*$-class.*

Before we sketch the proof of Theorem 3.2, we shall show how to derive the Adams-Kechris Theorem.

PROOF OF THEOREM 3.1. As Theorem 2.1 dealt with the case when $n = 1$, we can suppose that $n \geq 2$. By Theorem 2.6, there exists an $SL_{n+1}(\mathbb{Z})$-invariant Borel subset $X_{n+1} \subseteq S^*(\mathbb{Q}^{n+1})$ with an invariant ergodic nonatomic probability measure μ. Suppose that $(\cong_{n+1}^*) \leq_B (\cong_n^*)$. Then there exists a Borel function $f : X_{n+1} \to S^*(\mathbb{Q}^n)$ such that $A \cong_{n+1}^* B$ iff $f(A) \cong_n^* f(B)$. If A is $E^{X_{n+1}}_{SL_{n+1}(\mathbb{Z})}$-equivalent to B, then $A \cong_{n+1}^* B$ and so $f(A) \cong_n^* f(B)$. Hence by Theorem 3.2, there exists an $SL_{n+1}(\mathbb{Z})$-invariant Borel subset $M \subseteq X_{n+1}$ with $\mu(M) = 1$ such that f maps M into a single $\cong_n^*$-class $\mathcal{C}$. But clearly $f^{-1}(\mathcal{C})$ consists of only countably many $E^{X_{n+1}}_{SL_{n+1}(\mathbb{Z})}$-classes, which contradicts the fact that μ is nonatomic. Hence $(\cong_{n+1}^*) \nleq_B (\cong_n^*)$. $\dashv$

Theorem 3.2 is a consequence of Zimmer's superrigidity theorems for cocycles [27], which were designed to study orbit equivalence of free measure preserving actions of linear algebraic groups. Roughly speaking, Zimmer

[27] used his cocycle superrigidity theorems to show that, under suitable hypotheses, if X is a standard Borel G-space with an invariant ergodic probability measure μ and G acts freely on X, then both the group G and its action on X are "almost completely determined" by the orbit equivalence relation E_G^X and the measure μ. (Strictly speaking, Zimmer's results are not quite this strong; but this is an accurate summary of the later work of Furman [12].) It was an important insight of Adams-Kechris [2] that Zimmer's cocycle superrigidity theorems could also be applied in a purely descriptive set-theoretic context.

In order to slightly simplify the various definitions, we shall restrict our discussion of cocycles to the case when both groups are countable. However, we should point out that cocycles are most naturally studied in the wider context of locally compact second countable groups. For example, while the statement of Theorem 3.5 only mentions countable groups, the proof involves algebraic groups over $\mathbb{R}$, $\mathbb{C}$ and various finite extensions of the p-adic fields $\mathbb{Q}_p$. (Clear accounts of the theory of cocycles can be found in Zimmer [27] and Adams-Kechris [2]. In particular, Adams-Kechris [2, Section 2] provides a convenient introduction to the basic techniques and results in this area, written for the non-expert in the ergodic theory of groups.)

Let G be a countable group and let X be a standard Borel G-space with an invariant probability measure μ.

DEFINITION 3.3. Suppose that H is a countable group. Then a Borel function $\alpha\colon G \times X \to H$ is called a *cocycle* if for all $g, h \in G$ and $x \in X$,

$$\alpha(hg, x) = \alpha(h, g.x)\alpha(g, x).$$

Cocycles typically arise in the following manner. Suppose that Y is a standard Borel H-space and that H acts freely on Y. If $f\colon X \to Y$ is a Borel function such that $xE_G^X y$ implies $f(x)E_H^Y f(y)$, then we can define a Borel cocycle $\alpha\colon G \times X \to H$ by letting $\alpha(g, x)$ be the *unique* element of H such that

$$\alpha(g, x).f(x) = f(g.x).$$

Suppose now that $B\colon X \to H$ is a Borel function and that $f'\colon X \to Y$ is defined by $f'(x) = B(x).f(x)$. Then $xE_G^X y$ also implies that $f'(x)E_H^Y f'(y)$; and the corresponding cocycle $\beta\colon G \times X \to H$ satisfies

$$\beta(g, x) = B(g.x)\alpha(g, x)B(x)^{-1}$$

for all $g \in G$ and $x \in X$. This observation motivates the following definition.

DEFINITION 3.4. Suppose that H is a countable group. Then the cocycles $\alpha, \beta\colon G \times X \to H$ are *equivalent*, written $\alpha \sim \beta$, iff there exist a Borel function $B\colon X \to H$ and a G-invariant Borel subset $X_0 \subseteq X$ with $\mu(X_0) = 1$ such that

$$\beta(g, x) = B(g.x)\alpha(g, x)B(x)^{-1}$$

for all $g \in G$ and $x \in X_0$.

The proof of Theorem 3.2 is based upon a suitable cocycle reduction result; i.e., a theorem which say that under appropriate hypotheses on G and H, every cocycle $\alpha \colon G \times X \to H$ is equivalent to a cocycle β such that $\beta(G \times X)$ is contained in a finite subgroup of H. To see why such a result might be useful, we shall continue our discussion of the example which was introduced just before Definition 3.4. Suppose now that $\alpha \sim \beta$, where $\beta(G \times X)$ is contained in the finite subgroup K of H. Let $B \colon X \to H$ and $X_0 \subseteq X$ be as in Definition 3.4; and let $f' \colon X \to Y$ be the Borel function defined by $f'(x) = B(x).f(x)$. Then for each $x \in X_0$,

$$\begin{aligned}\Phi(x) &= \{\beta(g,x).f'(x) \mid g \in G\}\\ &= \{f'(z) \mid z E_G^X x\}\end{aligned}$$

is a nonempty finite subset of Y; and clearly if xEy, then $\Phi(x) = \Phi(y)$. In other words, Φ is a G-invariant Borel map into the standard Borel space $Y^{<\omega}$ of all finite subsets of Y. Hence if μ is an ergodic measure on X, then there exists a G-invariant Borel subset $M \subseteq X_0$ with $\mu(M) = 1$ such that $\Phi \restriction M$ is a constant function; and this implies that f maps M into a single E_H^Y-class.

Before we state our main cocycle reduction theorem, we need to review some notions from the theory of algebraic groups. Throughout this paper, let Ω be a fixed algebraically closed field of characteristic 0 which contains $\mathbb{R}$ and all of the p-adic fields $\mathbb{Q}_p$. By an *algebraic group*, we shall always mean a subgroup G of some general linear group $GL_n(\Omega)$ which is Zariski closed in $GL_n(\Omega)$. If G can be defined using polynomials with coefficients in the subfield k of Ω, then we say that G is an algebraic k-group. If G is an algebraic k-group, then $G(k)$ denotes the group of all matrices in G with entries in k.

THEOREM 3.5 ([23]). *Let $m \geq 3$ and let X be a standard Borel $SL_m(\mathbb{Z})$-space with an invariant ergodic probability measure. Suppose that G is an algebraic $\mathbb{Q}$-group such that* $\dim G < m^2 - 1$ *and that $H \leqslant G(\mathbb{Q})$. Then for every Borel cocycle $\alpha \colon SL_m(\mathbb{Z}) \times X \to H$, there exists an equivalent cocycle β such that $\beta(SL_m(\mathbb{Z}) \times X)$ is contained in a finite subgroup of H.*

Theorem 3.5 is a straightforward consequence of Zimmer's superrigidity theorem [27, Theorem 5.2.5] and the ideas of Adams-Kechris [2].

PROOF OF THEOREM 3.2. Let $m \geq 3$ and let X be a standard Borel $SL_m(\mathbb{Z})$-space with an invariant ergodic probability measure μ. Let $1 \leq n < m$ and suppose that $f : X \to S^*(\mathbb{Q}^n)$ is a Borel function such that $x E_{SL_m(\mathbb{Z})}^X y$ implies $f(x) \cong_n^* f(y)$. Without loss of generality, we can suppose that $f(x)$ has rank n for each $x \in X$. This implies that the stabiliser of each group $f(x)$ in $GL_n(\mathbb{Q})$ is $\mathrm{Aut}(f(x)) = \{\pm I\}$; and so $\cong_n^* \restriction f(X)$ is induced by the corresponding free action of $H = GL_n(\mathbb{Q})/\{\pm I\}$ on the

Borel set $\{A \in S^*(\mathbb{Q}^n) \mid A \text{ has rank } n\}$. Hence we can define a Borel cocycle $\alpha\colon SL_m(\mathbb{Z}) \times X \to H$ by

$$\alpha(g, x) = \text{the unique } \varphi \in H \text{ such that } \varphi.f(x) = f(g.x).$$

Applying Theorem 3.5, there exists an equivalent cocycle $\beta \sim \alpha$ such that $\beta(SL_m(\mathbb{Z}) \times X)$ is contained in a finite subgroup of H. Arguing as in the paragraph after Definition 3.4, it follows that there exists an $SL_m(\mathbb{Z})$-invariant Borel subset M with $\mu(M) = 1$ such that f maps M into a single $\cong_n^*$-class. $\dashv$

In the next section, we shall also make use of the following cocycle reduction result, which is a special case of Hjorth-Kechris [15, Theorem 10.5]. (While there is no explicit mention of cocycles in the statement of Theorem 3.6, the proof ultimately relies upon a cocycle reduction theorem of Adams-Spatzier [3].)

THEOREM 3.6 ([15]). *Let $m \geq 3$ and let X be a standard Borel $SL_m(\mathbb{Z})$-space with an invariant ergodic probability measure μ. Suppose that Y is a standard Borel space and that F is a hyperfinite equivalence relation on Y. If $f\colon X \to Y$ is a Borel function such that $xE^X_{SL_m(\mathbb{Z})}y$ implies $f(x)Ff(y)$, then there exists an $SL_m(\mathbb{Z})$-invariant Borel subset M with $\mu(M) = 1$ such that f maps M into a single F-class.*

§4. Torsion-free groups of finite rank.

In this section, we shall sketch the proof that $(\cong_n) <_B (\cong_{n+1})$ for all $n \geq 1$. Arguing as in the proof of Theorem 3.1, it is enough to prove the following result. (Instead of working directly with $S(\mathbb{Q}^n)$, we shall find it slightly more convenient to work with the Borel subset $R(\mathbb{Q}^n)$ consisting of those subgroups $A \in S(\mathbb{Q}^n)$ which have rank exactly n.)

THEOREM 4.1 ([23]). *Let $m \geq 3$ and let X be a standard Borel $SL_m(\mathbb{Z})$-space with an invariant ergodic probability measure μ. Suppose that $1 \leq n < m$ and that $f\colon X \to R(\mathbb{Q}^n)$ is a Borel function such that $xE^X_{SL_m(\mathbb{Z})}y$ implies $f(x) \cong_n f(y)$. Then there exists an $SL_m(\mathbb{Z})$-invariant Borel subset M with $\mu(M) = 1$ such that f maps M into a single $\cong_n$-class.*

We shall begin by saying a few words about the strategy of the proof of Theorem 4.1. So suppose that $m \geq 3$ and that X is a standard Borel $SL_m(\mathbb{Z})$-space with an invariant ergodic probability measure μ. Suppose that $1 \leq n < m$ and that $f\colon X \to R(\mathbb{Q}^n)$ is a Borel function such that $xE^X_{SL_m(\mathbb{Z})}y$ implies $f(x) \cong_n f(y)$. As we pointed out in Section 2, the action of $GL_n(\mathbb{Q})$ on $R(\mathbb{Q}^n)$ is far from free. In fact, for each $A \in R(\mathbb{Q}^n)$, the stabiliser of A in $GL_n(\mathbb{Q})$ is precisely the automorphism group $\mathrm{Aut}(A)$ of A. Consequently, we are initially unable to define a corresponding cocycle $\alpha\colon SL_m(\mathbb{Z}) \times X \to GL_n(\mathbb{Q})$.

In Thomas [22], we were able to get around this difficulty, in the case when $n = 2$, by reducing to the situation where there exists a Borel subset $X_0 \subseteq X$ with $\mu(X_0) = 1$ such that $\mathrm{Aut}(f(x))$ is a *fixed* subgroup L of $GL_2(\mathbb{Q})$ for all $x \in X_0$. This meant that $\cong_2 \restriction f(X_0)$ was induced by a free action of the quotient group

$$H = N_{GL_2(\mathbb{Q})}(L)/L$$

on the Borel subset $\{A \in R(\mathbb{Q}^2) \mid \mathrm{Aut}(A) = L\}$ of $R(\mathbb{Q}^2)$; and so we could define a corresponding cocycle $\alpha\colon SL_m(\mathbb{Z}) \times X \to H$. However, this would not have been useful unless there existed a suitable cocycle reduction result for cocycles taking values in H; and for such a result to exist, it was necessary that H should be a "reasonably classical" group. Fortunately, using Król's analysis [19] of the automorphism groups of the torsion-free abelian groups of rank 2, we were able to explicitly enumerate the possibilities for L; and it turned out that we could make a further reduction to the situation when H was a slight variant of $PGL_2(\mathbb{Q})$.

Now suppose that $n > 2$. Then it seems likely that we can once again reduce to the situation where there exists a Borel subset $X_0 \subseteq X$ with $\mu(X_0) = 1$ such that $\mathrm{Aut}(f(x))$ is a fixed subgroup L of $GL_n(\mathbb{Q})$ for all $x \in X_0$. Unfortunately, as n gets larger, the possibilities for L become more and more complex. In fact, by Corner's Theorem [7], L can be isomorphic to the group of units of an arbitrary reduced subring of $\mathrm{Mat}_d(\mathbb{Q})$, where $d = \lfloor n/2 \rfloor$; and it is far from clear whether $H = N_{GL_n(\mathbb{Q})}(L)/L$ will always be a "reasonably classical" group. (Here a ring is said to be *reduced* iff its additive subgroup contains no nontrivial divisible subgroups.) In order to avoid these algebraic complications, we shall initially replace the isomorphism relation on $R(\mathbb{Q}^n)$ by the coarser relation of quasi-isomorphism. This relation was first introduced in Jónsson [17], where it was shown that the class of torsion-free abelian groups of finite rank has a better decomposition theory with respect to quasi-isomorphism than with respect to isomorphism. This decomposition theory will not concern us. Rather we shall exploit the fact that much of the number-theoretical complexity of a finite rank torsion-free abelian group is lost when we work with respect to quasi-isomorphism; and this turns out to be enough to ensure that the cocycles that arise in our analysis always take values in a "reasonably classical" group. As a bonus, we obtain that the problem of classifying the torsion-free abelian groups of finite rank up to quasi-isomorphism is also intractible.

DEFINITION 4.2. Suppose that $A, B \in R(\mathbb{Q}^n)$. Then A is said to be *quasi-contained* in B, written $A \prec_n B$, if there exists an integer $m > 0$ such that $mA \leqslant B$. If $A \prec_n B$ and $B \prec_n A$, then A and B are said to be *quasi-equal* and we write $A \approx_n B$.

It is well-known that if $A \in R(\mathbb{Q}^n)$ and $m > 0$, then $[A\colon mA] < \infty$. This implies that if $A, B \in R(\mathbb{Q}^n)$, then $A \approx_n B$ iff $A \cap B$ has finite index in both A

and B. It is an easy exercise to show that $\approx_n$ is a countable Borel equivalence relation on $R(\mathbb{Q}^n)$. The next result seems to be more difficult. (The proof in [23] makes use of the Kurosh-Malcev [20, 21] p-adic localisation technique.)

THEOREM 4.3 ([23]). *For each $n \geq 1$, the relation $\approx_n$ is hyperfinite.*

DEFINITION 4.4. Suppose that A, $B \in R(\mathbb{Q}^n)$. Then A and B are said to be *quasi-isomorphic*, written $A \sim_n B$, if there exists $\varphi \in GL_n(\mathbb{Q})$ such that $\varphi(A) \approx_n B$.

Most of our effort in this section will be devoted to proving the following analogue of Theorem 4.1 for the quasi-isomorphism relation.

THEOREM 4.5 ([23]). *Let $m \geq 3$ and let X be a standard Borel $SL_m(\mathbb{Z})$-space with an invariant ergodic probability measure μ. Suppose that $1 \leq n < m$ and that $f : X \to R(\mathbb{Q}^n)$ is a Borel function such that $xE^X_{SL_m(\mathbb{Z})}y$ implies $f(x) \sim_n f(y)$. Then there exists an $SL_m(\mathbb{Z})$-invariant Borel subset M with $\mu(M) = 1$ such that f maps M into a single $\sim_n$-class.*

Arguing as in the proof of Theorem 3.1, we now easily obtain the following result.

THEOREM 4.6 ([23]). $(\sim_n) <_B (\sim_{n+1})$ *for all $n \geq 1$.*

Before we present the proof of Theorem 4.5, we shall show how to derive Theorem 4.1.

PROOF OF THEOREM 4.1. Suppose that $f : X \to R(\mathbb{Q}^n)$ is a Borel function such that $xE^X_{SL_m(\mathbb{Z})}y$ implies $f(x) \cong_n f(y)$. Then obviously $xE^X_{SL_m(\mathbb{Z})}y$ implies $f(x) \sim_n f(y)$; and so by Theorem 4.5, there exists an $SL_m(\mathbb{Z})$-invariant Borel subset Y with $\mu(Y) = 1$ such that f maps Y into a single $\sim_n$-class $\mathcal{C}$. Since $\mathcal{C}$ is countable, there exists a Borel subset $Z \subseteq Y$ with $\mu(Z) > 0$ and a fixed group $A \in \mathcal{C}$ such that $f(x) = A$ for all $x \in Z$. Since μ is ergodic, the $SL_m(\mathbb{Z})$-invariant Borel subset $M = SL_m(\mathbb{Z}).Z$ satisfies $\mu(M) = 1$; and clearly f maps M into the $\cong_n$-class containing A. $\dashv$

For each $A \in R(\mathbb{Q}^n)$, let $[A]$ be the $\approx_n$-class containing A. During the proof of Theorem 4.5, we shall consider the action of $GL_n(\mathbb{Q})$ on the set of $\approx_n$-classes. In order to describe the setwise stabiliser in $GL_n(\mathbb{Q})$ of a $\approx_n$-class $[A]$, it is necessary to introduce the notions of a quasi-endomorphism and a quasi-automorphism. If $A \in R(\mathbb{Q}^n)$, then a linear transformation $\varphi \in \mathrm{Mat}_n(\mathbb{Q})$ is said to be a *quasi-endomorphism* of A iff $\varphi(A) \prec_n A$. Equivalently, φ is a quasi-endomorphism of A iff there exists an integer $m > 0$ such that $m\varphi \in \mathrm{End}(A)$. It is easily checked that the collection $\mathrm{QE}(A)$ of quasi-endomorphisms of A is a $\mathbb{Q}$-subalgebra of $\mathrm{Mat}_n(\mathbb{Q})$ and that if $A \approx_n B$, then $\mathrm{QE}(A) = \mathrm{QE}(B)$. A linear transformation $\varphi \in \mathrm{Mat}_n(\mathbb{Q})$ is said to be a *quasi-automorphism* of A iff φ is a unit of the $\mathbb{Q}$-algebra $\mathrm{QE}(A)$. The group of quasi-automorphisms of A is denoted by $\mathrm{QAut}(A)$.

LEMMA 4.7 ([23]). *If $A \in R(\mathbb{Q}^n)$, then* $\mathrm{QAut}(A)$ *is the setwise stabiliser of $[A]$ in $GL_n(\mathbb{Q})$.*

Now we are ready to present the proof of Theorem 4.5. So let $m \geq 3$ and let X be a standard Borel $SL_m(\mathbb{Z})$-space with an invariant ergodic probability measure μ. Suppose that $1 \leq n < m$ and that $f : X \to R(\mathbb{Q}^n)$ is a Borel function such that $xE^X_{SL_m(\mathbb{Z})}y$ implies $f(x) \sim_n f(y)$. Let $E = E^X_{SL_m(\mathbb{Z})}$ and for each $x \in X$, let $A_x = f(x) \in R(\mathbb{Q}^n)$. First notice that there are only countably possibilities for the $\mathbb{Q}$-algebra $\mathrm{QE}(A_x)$. Hence there exists a Borel subset $X_1 \subseteq X$ with $\mu(X_1) > 0$ and a fixed $\mathbb{Q}$-subalgebra S of $\mathrm{Mat}_n(\mathbb{Q})$ such that $\mathrm{QE}(A_x) = S$ for all $x \in X_1$. By the ergodicity of μ, we have that $\mu(SL_m(\mathbb{Z}).X_1) = 1$. In order to simplify notation, we shall assume that $SL_m(\mathbb{Z}).X_1 = X$. After slightly adjusting f if necessary, we can suppose that $\mathrm{QE}(A_x) = S$ for all $x \in X$. (More precisely, let $c : X \to X$ be a Borel function such that $c(x)Ex$ and $c(x) \in X_1$ for all $x \in X$. Then we can replace f with $f' = f \circ c$.) In particular, we have that $\mathrm{QAut}(A_x) = S^*$, the group of units of S, for each $x \in X$. Now suppose that $x, y \in X$ and that xEy. Then $A_x \sim_n A_y$ and so there exists $\varphi \in GL_n(\mathbb{Q})$ such that $\varphi(A_x) \approx_n A_y$. Notice that

$$\varphi S \varphi^{-1} = \varphi\, \mathrm{QE}(A_x)\varphi^{-1} = \mathrm{QE}(\varphi(A_x)) = \mathrm{QE}(A_y) = S$$

and so $\varphi \in N = N_{GL_n(\mathbb{Q})}(S)$. Clearly we also have that $\varphi([A_x]) = [A_y]$; and by Lemma 4.7, for each $x \in X$, the stabiliser of $[A_x]$ in $GL_n(\mathbb{Q})$ is $\mathrm{QAut}(A_x) = S^*$. Let $H = N/S^*$ and for each $\varphi \in N$, let $\overline{\varphi} = \varphi S^*$. Then we can define a Borel cocycle $\alpha : SL_m(\mathbb{Z}) \times X \to H$ by

$$\alpha(g, x) = \text{the unique element } \overline{\varphi} \in H \text{ such that } \varphi([A_x]) = [A_{g.x}].$$

LEMMA 4.8 ([23]). *There exists an algebraic* $\mathbb{Q}$*-group* G *with* $H \leqslant G(\mathbb{Q})$ *such that* $\dim G < m^2 - 1$.

PROOF. Recall that Ω denotes a fixed algebraically closed field containing $\mathbb{R}$ and all of the p-adic fields $\mathbb{Q}_p$. Let

$$\Lambda = \Omega \otimes S \subseteq \mathrm{Mat}_n(\Omega)$$

be the associated Ω-algebra. Then Λ is an affine $\mathbb{Q}$-variety; and the Cayley-Hamilton Theorem implies that the group of units of Λ is given by

$$\Lambda^* = \{\varphi \in \Lambda \mid \det(\varphi) \neq 0\}.$$

Thus Λ^* is an algebraic $\mathbb{Q}$-group and $\Lambda^*(\mathbb{Q}) = S^*$. Furthermore, by Proposition 1.7 [6], $\Gamma = N_{GL_n(\Omega)}(\Lambda)$ is also an algebraic $\mathbb{Q}$-group and clearly $\Gamma(\mathbb{Q}) = N$. By Theorem 6.8 [6], $G = \Gamma/\Lambda^*$ is an algebraic $\mathbb{Q}$-group and

$$H = \Gamma(\mathbb{Q})/\Lambda^*(\mathbb{Q}) \leqslant G(\mathbb{Q}).$$

Finally note that

$$\dim G \leq \dim \Gamma \leq \dim GL_n(\Omega) = n^2 < m^2 - 1. \qquad \dashv$$

By Theorem 3.5, α is equivalent to a cocycle β such that $\beta(SL_m(\mathbb{Z}) \times X)$ is contained in a finite subgroup K of H. To simplify notation, we shall assume that $\beta = \alpha$. (Recall from Section 3 that there exists a Borel function $B\colon X \to H$ such that on an $SL_m(\mathbb{Z})$-invariant Borel subset $X_0 \subseteq X$ with $\mu(X_0) = 1$, the cocycle β corresponds to the Borel function $f'\colon X \to R(\mathbb{Q}^n)$ defined by $f'(x) = B(x).f(x)$.) Then for each $x \in X$,

$$\begin{aligned}\Phi(x) &= \{\varphi([A_x]) \mid \overline{\varphi} = \alpha(g, x) \text{ for some } g \in G\}\\ &= \{[A_z] \mid zEx\}\end{aligned}$$

is a nonempty finite set of $\approx_n$-classes; and clearly if xEy, then $\Phi(x) = \Phi(y)$. By the ergodicity of μ, we can suppose that there exists an integer $k \leq |K|$ such that $|\Phi(x)| = k$ for all $x \in X$. Now let $x \mapsto (x_1, \dots, x_k)$ be a Borel function from X to X^k such that for each $x \in X$,

(a) $x_i E x$ for all $1 \leq i \leq k$; and
(b) $\Phi(x) = \{[A_{x_1}], \dots, [A_{x_k}]\}$.

Finally let $\widetilde{f}\colon X \to R(\mathbb{Q}^n)^k$ be the Borel function defined by

$$\widetilde{f}(x) = (A_{x_1}, \dots, A_{x_k});$$

and let F be the countable Borel equivalence relation on $R(\mathbb{Q})^k$ defined by

$$(A_1, \dots, A_k)\, F\, (B_1, \dots, B_k) \quad \text{iff} \quad \{[A_1], \dots, [A_k]\} = \{[B_1], \dots, [B_k]\}.$$

Using the fact that $\approx_n$ is hyperfinite, it follows easily that F is also hyperfinite. (For example, see Section 1 [16].) Notice that if xEy, then $\Phi(x) = \Phi(y)$ and so $\widetilde{f}(x) F \widetilde{f}(y)$. By Theorem 3.6, there exists an $SL_m(\mathbb{Z})$-invariant Borel subset $M \subseteq X$ with $\mu(M) = 1$ such that $\widetilde{f}$ maps M into a single F-class; and this implies that f maps M into a single $\sim_n$-class. This completes the proof of Theorem 4.5.

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