THE BULLETIN OF SYMBOLIC LOGIC
Volume 6, Number 1, March 2000

TOWARDS A RE-EVALUATION OF JULIUS KÖNIG'S CONTRIBUTION TO LOGIC

MIRIAM FRANCHELLA

Abstract. Julius König is famous for his mistaken attempt to demonstrate that the continuum hypothesis was false. It is also known that the only positive result that could have survived from his proof is the paradox which bears his name. Less famous is his 1914 book *Neue Grundlagen der Logik, Arithmetik und Mengenlehre*. Still, it contains original contributions to logic, like the concept of metatheory and the solution of paradoxes based on the refusal of the law of bivalence. We are going to discover them by analysing the content of the book.

§1. Introduction. Julius (Gyula) König represented a milestone for the progress of mathematics in Hungary due to his passionate work both for doing research and for up-dating educational programs in many branches of that discipline. He was born in Györ on 16th December 1849[1]. At the age of 16 he went to study at the Medical Department of the University of Vienna and at the same time attended lectures on mathematics. After a short stay in Vienna, he went to Heidelberg, where he got his doctoral degree in 1870 with a thesis on the theory of modular functions, although his first essay treated a medical subject (*Beiträge zur Theorie der elektrischen Nervenreizungen*). Then he spent half a year in Berlin, where he attended lectures of Weierstrass and Kronecker. He was appointed professor of the Teacher's College of Budapest in 1873 and professor of the Third Department of the Technical University one year later. He took part in all activities of this University. He was the dean of the faculty of engineering and architecture three times, and Rector of the University also three times. His teaching activity was not confined to the university. He held the view that the professors of the universities should always be in touch with secondary education, and they should guide and help work there. Hence, he established the education of commerce in Hungary and worked out the algebra part of the 1879 programme for secondary schools. He also wrote a textbook for it. He was

Received March 31, 1999; revised August 12, 1999.

[1]See Szénássy [1992, 217–223, 233–255, 332–335] to which we refer the reader for a complete bibliography of König.

1079-8986/00/0601-0002/$3.20

one of the founders of the *Mathematical Society* in Hungary and became a member of the Hungarian Academy of Sciences in 1889. In 1905, because of his manifold obligations, König retired, but he continued to lecture at the Technical University. He remained active until the last moment of his life: he was working on the last chapter of his monograph on the foundations of mathematics when he suddendly died on 8th April 1913. His son Dénes —the outstanding mathematician (he had another son, György, a doctor of law and historian of literature)— edited the book.

During his life Julius König contributed to many areas of mathematics: algebra, analysis, geometry, number theory, set theory. He devoted to algebra already his dissertation and about ten of his papers, mainly from his youth, dealing with elliptical functions applied in algebra, whose results reappeared later on in his 1903 monograph "Outlines of the general theory of algebraic quantities". He also wrote several papers on power series of the reciprocal $1/f(x)$ of a polynomial, and a textbook striving to improve more the students' way of mathematical thinking than their mere mathematical knowledge. As for analysis, he exerted his influence on Hungarian mathematical culture by his textbooks rather than his research papers. Two of his papers (on the theory of gamma functions and on the theory of definite integrals) were written for didactical purpose (simplifying proofs) while the others (mainly on the theory of differential equations) had more theoretical purposes and earned international acclaim. He contributed to number theory (with textbooks and) with a proof of the reciprocity theorem of quadratic residues and with a result put to paper by Gusztav Rados making its way into the literature as the König-Rados theorem (that establishes sufficient and necessary condition for a congruence mod p to have $p-1-r_p$ incongruent solutions). Furthermore, he published an early paper on geometry, where he presented how to obtain an intuitive description of three-dimensional non-Euclidean geometries (his model was rather an idea than an elaborated model). Abroad, he is mainly known for his interests in set theory, and in this field he is more famous for his mistaken attempt to demonstrate (at the Heidelberg Congress of Mathematics) that the continuum hypothesis was false and for the paradox which bears his name than for his 1914 book *Neue Grundlagen der Logik, Arithmetik und Mengenlehre*. This is only rarely considered in the literature: J. von Neumann affirmed [1927, 22] that Hilbert had got from König the idea of proving the noncontradictoriness of mathematics through an "evaluation" (*Wertung*); G.H. Moore gave some hints about the book [1982, 183–184] and specified [1982, 262] that it (instead of Zermelo's paper) seemed to be the source from which Mirimanoff took the axioms of union and power set. Still, the book offers many (other) aspects of originality for its time that deserve to be stressed: the concept of metatheory; the non-acceptance of the law of excluded middle among the axiomatized logical laws; the solution of paradoxes based on the refusal of the law of bivalence. We are going to consider them in detail in this paper.

§**2. König's proposals.** König had already anticipated in 1906[2] that he would write a detailed account of his reflections about the facts and laws underlying scientific thought. However, he could not finish the whole book: in particular, he did not complete the last section of the book and left only some fragments about the so-called "ordinators" (*Ordinatoren*). Still, he managed to express his main ideas.

Before considering in detail König's 1914 book it is useful to sketch here the situation of foundational research in that period. Logicism had appeared as a research program in all its aspects, positive and negative. Frege's *Grundgesetze der Arithmetik* had already been published while Russell/Whitehead's *Principia Mathematica* were published between 1910 and 1913. Many antinomies had been discovered, in addition to the very old liar paradox. We recall here only the Russell, the Berry-Dixon, the Burali-Forti paradox, that were directly considered by König, together with the paradox bearing his name, very similar to the Berry-Dixon. Intuitionism was not yet known as L.E.J. Brouwer had written his 1907 thesis "On the foundations of mathematics", his 1908 paper against the acceptability of the principle of excluded middle "The unreliability of logical principles" and his 1912 inaugural lecture "Intuitionism and formalism" only in Dutch and published them in Dutch journals. Only what Brouwer later called pre-intuitionism (or old intuitionism), best represented and made explicit by Henri Poincaré, was known. Formalism was —so to say— at its very beginning. Hilbert had already explained the new concept of axiomatics —with the connected requirements for a proof of independence, completeness and consistency— and planned ("Über die Grundlagen der Logik und der Arithmetik" of 1905) a program of a progressive and parallel construction of logic and mathematics (in its various branches) as formal systems, each to be founded by showing its consistency. Still, his theorization of metamathematics —finitistic mathematics has not yet been conceived. In the meanwhile, at Hilbert's Göttingen[3], Zermelo in 1904 had proved that every cardinal number was an aleph, by using (and stating explicitly) the axiom of choice, and in 1908 had presented his axiomatization of set theory: he started with a domain (*Bereich*) D of things —some of which are sets— and their fundamental relationships: "$a = b$" and "$a \in b$" (a belongs to b). Then, axioms are posited that state the existence of some sets (elementary sets and an infinite set) and establish the condition for their equality (extensionality axiom). Finally, other axioms fix the admissible operations on sets that generate further sets: the power set axiom, the sum set axiom, the choice axiom and the *Aussonderungsaxiom*.

[2][1906, 217].

[3]On this subject see Peckhaus [1990].

The last of these establishes that a subset can be formed from another set by a property P, provided that this is *definit*, i.e. such that "for each single element x of the starting set the fundamental relationships of the axiom system decide, via axioms and generally valid logical laws, in a not arbitrary way whether P holds or not for x"[4]. Thanks to the two conditions (one has to start with a set and the property should be *definit*) required and the fact that the domain is proved not to be a set, all antinomies disappear. It has been discussed in the literature whether solving antinomies was the main aim of Zermelo's axiomatization of set theory. In particular, G.H. Moore (1978) and later (1984) M. Hallett sustained, against all the preceding interpretations, that such axiomatization was primarily motivated by a desire to save his axiom of choice and the related proof of the well-ordering theorem. In any case, the first occasion for a meeting between König and Zermelo was given by the axiom of choice. Namely, König's book was written at the end of a series of (failing) contributions to set theory which aimed to show that the continuum hypothesis was false since the cardinality of the continuum was not an aleph. At the very beginning, König strongly refused the axiom of choice, but, after that, Zermelo found the error in the proof during the 1904 Heidelberg congress and König later converted to Zermelo's idea. As we are going to see, in his book König accepted Zermelo's axiom of choice.

The starting point of König's book was his belief that at the basis of both mathematics and logic there was mental evidence. He called its description "synthetic logic" (*synthetische Logik*) and specified[5] [1914, iii] that this should have the same relationship with thought as mechanics has with planets' movements. He set synthetic logic as the task of his book, adding that he also obtained in it that[6] "*the strong requirements put by Poincaré for our mathematical and logical thought were satisfied and nevertheless classical results of Cantor's set theory were preserved*" [1914, iii]. We shall see at the end of the article in which sense König obtained this. As for evidence, he defined it [1914, 162] as[7] "*the feeling of the identity of the thought with itself*" and stressed that "evidences" are "irrefusable experiences of consciousness". The concept of irrefusability reminds us of Descartes' tradition of defining

[4]"Eine Frage oder Aussage P heißt definit für die Elemente der Menge m, wenn für jedes einzelne Element x von m die Grundbeziehungen der Axiomatik vermöge der Axiome und den allgemeingültigen logischen Gesetzen ohne Willkür entscheiden ob P für x gilt oder nicht" [1908, 262].

[5]"Die synthetische Logik soll ein wissenschaftliches Bild der in der Logik, Arithmetik und Mengenlehre angeschauten Denkvorgänge bieten, genau so wie die Mechanik des Himmels ein wissenschaftliches Bild der Planetenbewegung liefert".

[6]"[...] die von Poincaré geforderten strengen Bedingungen für unser mathematisches und logisches Denken erfüllt werden, und trotzdem die klassischen Resultate der Cantorschen Mengenlehre erhalten bleiben".

[7]"Das Gefühl der Identität meines Denkens mit sich selbst".

evidence as "*what one cannot refuse own's assent to*"[8], but König did not mention any author and used for it the term *Erlebnis* that was very common at that time[9]. However König, in a footnote [1914, 105], concerning necessity (which is the same as non-refusability for him) quoted Sigwart's "Logik", where [1904, 15] this author affirmed that man is capable of distinguishing between necessary and not- necessary thought.

He realised that consciousness is constituted by various states —so to say— that he called in German *Denkbereiche*. As mathematics has its foundations in consciousness, mathematical objects must correspond to some irrefusable experiences in a *Denkbereich*. König defined [1914, 15] a *Denkbereich* as the representation of a state of consciousness such that one is able, for any *Erlebnis*, to establish whether or not it belongs to this state. Yet, he added that also representations of states of consciousness about which one cannot establish for any *Erlebnis* either that it belongs to the *Denkbereich* or that it does not, are "*useful as objects of intuition*" ("*können als Objekte unserer Anschauung dienen*"). Two *Erlebnisse* can be either the repetition of each other[10] or different. Each *Erlebnis* is linked to a representation which is used as its name (also the representation of an *Erlebnis* is an *Erlebnis*). The name of a representation (used as a name) is a sign.

Since in most cases we have a partial overview of a *Denkbereich*, symbolic systems and logical deduction are useful, because they help us with finding further information about *Denkbereiche*. By describing the *Denkbereiche*, the so-called *Grundnorm* (fundamental rule) must be fulfilled: the same *Erlebnis* cannot be considered at the same time different and not-different, otherwise the *Denkbereich* is impossible[11]. The *Grundnorm* is considered by König as a kind of limitation upon our thought.

§3. Sets. The mathematical entities for which König found a mental counterpart were the following: classes, pure sets, usual sets, representations, logical forms, pure logic, axiomatic theories.

Classes were *Denkbereiche* where the same thing (determined by the proper name X) was given to every *Erlebnis* as a name, and the *Erlebnisse* could not be different from each other [1914, 24–32].

Pure sets were *Denkbereiche* where the same thing (determined by the proper name X) was given to every *Erlebnis* but the *Erlebnisse* could be different from each other and are supposed to be distinguishable from each other [1914, 33–36]. The way was specified (called α-way, β-way and so

[8]See *Meditationes* (*Oeuvres* VII, pp. 58–59).

[9]See Ritter [1992, 708].

[10]Since they are psychological states, two of them cannot be "the same" but at most "the repetition" of each other.

[11]The concept of contradictoriness appears here: hence, according to König, the concept of difference comes before that of contradictoriness.

on) in which the attribution was done. In symbols "$a \, \mathrm{rel}_\alpha \, X$". In other words, König considered from an abstract viewpoint the formation of a set X according to a certain property P and expressed this fact as the attribution of the thing X to (not specified but supposed distinguishable) things so that they are linked by the relationship α (determined by the property P).

Usual sets (=Cantorian sets) were *Denkbereiche* where it was specified that the *Erlebnisse* were *given* and then the same thing was attributed to them [1914, 37]. This is a complicated and quite obscure description by König. Still, we can try to understand it better, by considering the symbolic expression that König gave for it:

$$[x \text{ qual } G] \operatorname{inv}[x \, \mathrm{rel}_\alpha \, M],$$

which has to be read as "that x has the property G *involviert* that x is in the relationship α with M", where an "*Involution*" is a statement according to which, with A, also B is an *Erlebnis* belonging to the *Denkbereich* [1914, 19]. Hence, König wanted to intend that the fact that there are some objects with a certain property G[12] and the fact that the same thing is attributed to all of them on the basis of that property are *both* considered: this meant collecting existing things. Therefore, König affirmed that they are the sets usually treated in set-theory and called them "Cantorian sets". He also stressed that the thing attributed to the things is determined during the act of attributing it to the things.

In practice, classes are simply a way of naming things (that is why the things considered are equal); pure sets are the most abstract way of considering sets, that is by concentrating on the fact that they are an act of attribution (of a thing); usual sets are the standard way of conceptualizing sets, by stressing both the act of attribution and the existent objects towards which the attribution is directed. Pure sets collect things sharing a property; usual sets on the one side have things sharing a property and on the other side link them according to it. Hence, usual sets identify themselves with the things belonging to them. In each case, König stated the possibility that the thing that is attributed to the other can be attributed to itself: in this case we have sets of "second kind" (as an example, for the case of usual sets, we can consider the set of things that are not-red: such set can be attributed to itself as it possesses the property of being not-red).

König observed that (Cantorian) sets also had another supposed property: being noncontradictory. This is generally in contrast with the existence of paradoxes, hence different solutions have been proposed in order to avoid them. The first one quoted (in a footnote [1914, 39]) by König was Schoenflies' solution, which consisted in prohibiting self-reference of sets in any case. König disagreed with this position that seemed to him too strong and

[12]König underlined [1914, 41] that the property must be *definit* in the sense of Zermelo that we have recalled in §2.

limiting. Namely, he admitted for both pure and usual sets the possibility of attributing the set to itself. As for avoiding paradoxes, he maintained, instead, that it was enough (and necessary) to check directly for each new set that its definition (i.e. the specification of the domain G and the relationship α in the general scheme of definitions given above) respected the *Grundnorm* of thought. Since, in the general definition of usual sets, two *Denkbereiche* are involved (G and M), in order to establish if a single definition is admissible, one has to prove that both G and M are possible (do not give place to contradictions). If G is impossible (i.e., if the related *Erlebnisse* are impossible, if there are not such existing things), König said that the set did not exist; if G is possible but M is not (i.e., if it is impossible to collect the existing things), König said that the set has no logical existence.

It is *a priori* sure that G is possible:

(1) when "x qual G" does not mean that "the *Erlebnis* x does not belong to a certain *Denkbereich*";
(1) when, in the description of G, the α-relation is not used (in this case the first condition cannot be fulfilled either).

§**4. Logical forms.** Logical forms are introduced in order to give a logical system which can be considered as the basis of any further axiomatic theory. They are the result of the (finite) application of some rules to the basic logical forms (i.e., to logical axioms) and concern only propositional logic. The very interesting thing is the presentation of the following forms among the basic logical forms [1914, 99–100][13]:

Va. $[(x \sim\sim v) \sim\sim v] = [x \sim\sim v]$
Vb. $[(x \sim\sim v) \sim\sim v'] = [x \sim\sim v']$
Vc. $[(x \sim\sim v') \sim\sim v] = [x \sim\sim v']$
Vd. $[(x \sim\sim v') \sim\sim v'] = [x \sim\sim v]$
Ve. $[(x \cdot y) \sim\sim v] = [(x \sim\sim v) \cdot (y \sim\sim v)]$
Vf. $[(x \cdot y) \sim\sim v'] = [(x \sim\sim v') + (y \sim\sim v')]$
Vg. $[(x + y) \sim\sim v] = [(x \sim\sim v) + (y \sim\sim v)]$
Vh. $[(x + y) \sim\sim v'] = [(x \sim\sim v') \cdot (y \sim\sim v')]$[14].

They concern truth-values as $x \sim\sim v$ means that the proposition x has truth-value "true" and $x \sim\sim v'$ means that the proposition x has truth-value "false". König also gave a graphical representation of the attribution

[13] We have to specify here what "=" means for König. He called this relation [1914, 72–73] "isology" (*Isologie*), defined it as "non-diversity with respect to certain relationships established once for ever" so that he could include the case of identity (non-diversity with respect to every relationship), that he wanted to preserve but that represented for him an "obscure philosophical case" (it was not clear to him whether or not it was something realizable).

[14] "$x \cdot y$" is the expression for the logical product; "$x + y$" is the expression for the logical sum.

of truth-values as truth-tables (*Wertetafeln*), with two columns, one for v and one for v': propositions had to be put in either column, but there was also the possibility that a proposition could not be put in any column. He gave examples only of "basic" truth-tables, i.e. the following three:

I

v	v'
A	B

This represents the fact that A is true and B is false;

II

v	v'
A	B
$A \sim\sim v$	$A \sim\sim v'$
$B \sim\sim v'$	$B \sim\sim v$

This is an enlargment of the first table, where, on the basis of the truth of A and the falsity of B, it is expressed that the statement "A is true" and "B is false" are true while "A is false" and "B is true" are false.

III

v	v'
A	B
$A \sim\sim v$	$A \sim\sim v''$
$B \sim\sim v'$	$B \sim\sim v$
$(A \sim\sim v) \sim\sim v$	$(A \sim\sim v') \sim\sim v$
$(B \sim\sim v') \sim\sim v$	$(B \sim\sim v) \sim\sim v$
$(A \sim\sim v') \sim\sim v'$	$(A \sim\sim v) \sim\sim v'$
$(B \sim\sim v) \sim\sim v'$	$(B \sim\sim v') \sim\sim v'$

This is a further enlargment where also statements asserting the truth-value of statements asserting the truth-value of A and B are considered. Of course, it is possible to go on with these enlargements by considering statements asserting the truth-value of statements asserting the truth-value of statements asserting the truth-value of A and B, and building a table IV, and then so on. König also added that it is possible to build further tables considering also $A \cdot B$ and $A + B$ according to the rules for the attribution of truth-values put in the corresponding logical forms, but he did not write down the tables.

According to W. and M. Kneale [1962, 420], the method of presenting logical operations by comparing the possible truth-values can be traced back to Boole's *Mathematical Analysis of Logic* [1847, 49–50] and Frege's *Begriffsschrift* [1879, 5], although the term "truth-value" appeared only in Frege [Kneale 1962, 413]. A reference to tables for indicating combinations of truth-values appeared in Zermelo's lectures *Mathematische Logik* of 1908, according to the notes taken by K. Grelling and quoted by V. Peckhaus [1994, 319]. Truth-tables came definitively into use after Wittgenstein's 1921 "Logisch-philosophische Abhandlung" [4.31] and Post's "Introduction to a general theory of elementary propositions" of the same year. We can observe that, in König's work, rows and columns had an exchanged position with respect to these modern truth-tables. Furthermore, not only could propositions be put in any column, but also attributions of value to a proposition (for instance $x \sim\sim v$), attributions to attributions of value to a proposition (for instance $(x \sim\sim v) \sim\sim v'$), and so on. The fact that it was not established for any *Erlebnis* x that either $x \sim\sim v$ or $x \sim\sim v'$ held was well compatible with the fact stressed above that not for every *Erlebnis* is it decidable if it belongs to the *Denkbereich* or not. In this case, the non-decidable *Erlebnisse* are those which affirm "$x \sim\sim v$" and "$x \sim\sim v'$", respectively. König underlined that, according to him, it was impossible for everything to be determined in a *Denkbereich*, and that this was the main difference between his system and those presented by logical algebraists [1914, 87]. He did not mention any of them but only stressed this difference with respect to all of them in general, as an example of the wider and more substantial difference between his and their system (although they could appear to be similar in using a symbolic notation): while algebra of logic aimed only to a formalization, König aimed to produce new "methods of thinking" (*neue Denkmethoden*) that, on the one hand, could prevent any error that might accompany logical thought when expressed in its usual linguistic form, and, on the other hand, could provide new results (for instance, about the question of noncontradictoriness of formal systems).

§5. Pure logic and axiomatic systems. The *Denkbereich* of pure logic is obtained from that of basic logical forms, by interpreting its formulas, i.e. by attributing to them a truth-value.

At this point one should check if these descriptions are plausible, by controlling if the *Grundnorm* is respected. That is why König did not add the law of contradiction to basic logical forms: because he wanted to *check* that his logical system was noncontradictory. In order to do this, he divided the "forms" (=propositions) into two disjoint groups (incompatible to each other): P-forms and Q-forms (later, John von Neumann will call this method "evaluation"). Then he proved [1914, 114–120] by induction that logical forms could only be P-forms, hence that they did not bring a contradiction[15]. König firstly fixed for each connective and also for each kind of attribution of a logical value to a proposition (i.e. for the expressions "$x \sim\sim v$" and "$x \sim\sim v'$") whether they are P- or Q- forms. Secondly, he proved that basic logical forms are P-forms and that all rules for producing logical forms

[15] König did not prove every step, but gave only some examples, and then invited the reader to "experience" by himself all the other steps.

produce P-forms if they start from P-forms. The principle of contradiction gets its evidence from the practice of the procedure of consistency-proofs, but it is not a logical law to be inserted among the basic logical forms. It can be proved that the principle of excluded middle[16] is equivalent to it, therefore also this is evident but it is not a logical law. König underlined that, if it were to be added as a logical law (even under the weaker form: $(x \to y) \to (Fy \to Fx)$, where "F" means "false"), then one would obtain an absolute *Wahrheitsbereich*, that of traditional logic, that he considered as something artificial, because it excludes undecidable *Erlebnisse* which really exist.

The procedure of checking whether some forms are only P-forms can be applied to any axiomatic theory that is built by adding to fundamental logical forms some other basic forms (called "axiomatic forms") characterised by their decidability, i.e. by the fact that for any *Erlebnis* it is possible to establish whether or not it has this form. Once we establish that a certain axiomatic theory is not-contradictory, we can add the principle of contradiction to it, by obtaining another *Denkbereich*, "higher" with respect to that of the first axiomatic theory [1914, 141]: in this remark is contained the concept of "metatheory", although he did not introduce this term. In particular, we have seen the cases above in which a definition of set is acceptable. In these, we are allowed to add their definitions to the other axioms. An interesting case of this fact is König's acceptance of Zermelo's law of choice [1914, 162–172]. One starts from a "Zermelo set" N, i.e. from a set of sets. Then one can define the axiomatic form:

$$[X \operatorname{rel}_\alpha N] \operatorname{inv} [\operatorname{Prot} X \operatorname{rel}_\alpha AN]$$

where "Prot X" indicates one and only one element of X and "AN" indicates the "choice-set". Hence, the axiomatic form says that if X is (a set which is) an element of N, then one of its elements is an element of the choice-set, i.e., it is possible to form a choice-set by picking up an element from each set belonging to the starting set N. König stressed that this did not mean having a rule for choosing which element to pick up: this would be required from what he called a "principle of choice" (contrasted with the "law" of choice expressed by the above axiomatic form). He stated that there was evidence for the law of choice[17] (i.e., everybody can recognise that the *Denkbereich*

[16] According to a terminology very common in current literature, the "principle of excluded middle" must be interpreted as the law of bivalence, i.e. at a metalinguistic level, because for König negation means "judging as false".

[17] Not in every case it is so easy and immediate to establish that the a definition of a set is acceptable, in particular that the set is noncontradictory. Namely, for the empty set and for the set of all things he felt the need to prove the consistency of their axiomatic forms (respectively "$[a \operatorname{rel}_\alpha O \sim\sim v']$" for every a for the empty set and "a qual G" for every a for the set of all things), by observing that this becomes clear [1914, 151] if "x qual G" is attributed to the P-forms (true forms) for any x (as it is obvious according to its definition)

of the above axiomatic form is both possible and noncontradictory), but not for the principle of choice. This opinion is very different from what will be the common point of view of both constructivists and intuitionists. According to these, in fact, "evidence" will mean "effective constructability in thought", hence evidence requires a construction rule: that is why they will not accept the law of choice at all, since it would be acceptable (according to their criteria and using König's terminology) only as a principle of choice.

Finally, König gave an axiomatic system for arithmetic by adding as axiomatic forms the laws of definition for operations (sum, product, power) and proved its noncontradictoriness (by using the method of evaluation)[18].

§**6. Representations.** Representations are introduced by König [1914, 53–57] through the notion of relation. This is immediately acceptable since König had previously [1914, 5] noticed that "linking" is the first act of thought. A relation can be either "abstract" or "empirical". It is abstract when it is independent of the specification of the things which are linked. This kind of relation is expressed as "x is in an α-relation with y" (when the "α-relation" is specified). From abstract relations it is possible to build functions, which are expressed as "being in an α-relation with x". A case of function is the representation. Namely, it consists of starting from a thing x and building the set containing only x as element, a set that is called "the f-image of x" (briefly, $f(x)$). Hence, a representation is an example of function in the sense that the α-relation is "(being) the f-image". It is usual to express f-images through the symbols $1, 2, 3, \ldots$, in this way: 1 expresses $f(x)$; 2 expresses $ff(x)$, 3 expresses $fff(x)$, and so on. Thing and image are in a bijective correspondence to each other (given any of them, the other

and "$x \operatorname{rel}_\alpha O$" to the Q-forms (false forms) for any x (as it is obvious according to its definition). In this last case "$x \operatorname{rel}_\alpha O \sim\sim v'$" (stating the falsity of "$x \operatorname{rel}_\alpha O$") will be attributed to the P-forms for any x.

[18]König felt the need to credit [1914, 189] Hermann Grassmann as the first with the idea of developing arithmetic using the schema of induction and logical laws starting from certain "axioms". In fact, in his 1861 book *Lehrbuch der Arithmetik* Grassmann, dealing with the totality of all integers (positive, negative and 0) introduced recursive definitions for addition and multiplication, and proved on this basis ordinary laws of arithmetic by mathematical induction. Unlike König, Grassmann's work was not presented in an axiomatic form (although it can be easily put into an axiomatic form, as Hao Wang did in his 1957 article "The Axiomatization of Arithmetic"): that is why König put the expression "axioms", referring to Grassman's work, in inverted commas. Furthermore, Grassman did not explicitly mention the fact that different numbers have different successors, or the fact that 1 is not the successor of any positive integer. Hao Wang suggested as an explanation for this fact [1957, 149] that Grassmann took for granted that different numerals represented different objects. We recall here that Grassmann's *Lehrbuch der Arithmetik* was the main source for Schröder's 1873 *Lehrbuch der Arithmetik und Algebra*. For more information about this, see Peckhaus [1996]. Grassmann's work, together with that by Dedekind (in *Was sind und was sollen die Zahlen*) was at the origin of Giuseppe Peano's axiomatization of arithmetics. For more information about this, see Wang [1957].

is determined) and the possibility is supposed of reiterating the process, by producing f-f-images and so on. König himself explained the origin of this procedure. It derives from Zermelo regarding the idea of building singletons, and from Dedekind concerning the idea of forming chains: in practice, he started with Dedekind's chains but gave to $f(x)$ the meaning of "forming the singleton of x", according to Zermelo's building of set theory[19]. Furthermore, König noticed that it is impossible for our intuition to grasp a process of representation going on indefinitely. He also remarked [1914, 47] that it is impossible for our intuition to grasp representations that are finite but requiring a large number of steps (to build, for instance, one trillion). But this did not lead him to refuse both these kinds of representations: rather, he said that "high" and "indefinitely proceeding" reiterations of producing f-images should be admitted since each single step of their generating process is evident. In other words, these reiterations should be admitted because the capability of producing each of their steps is intrinsic to man: "*In this case we have to do not with a limit to our capacity of thinking but with a limit to our capacity of seeing*"[20] [1914, 47]. We cannot see the final result, but only each act of producing. This immediately recalls Poincaré's "répétition indéfinie d'un même acte dès que cet acte est une fois possible" [1902, 24].

Also empirical relations concern representations. Namely, empirical relations depend on the things that they link, which must therefore be specified (their expression is like "a is in the α-relation with b", where a and b are proper names). A first example of them is the "syllogistic relation", which is characterised by its transitivity (and in effect all syllogisms "work" due to transitivity). A clarifying case of syllogistic relation is the order relation (expressed as "$a < b$"), which is characterised not only by transitivity but also by the fact that between two things a and b only one of the relations: $a < b$ and $b < a$ can hold[21]. The order relation makes it clear why empirical relations depend on the linked things. In fact, (following König's example), if we first consider a set $A = \{a, b, c\}$ on which is defined the relation $R = \{a < b, a < c, b < c\}$, and then another set $A' = \{a, b, c, d\}$ on which is defined the relation $R' = \{a < b, a < c, b < c, a < d, b < d, c < d\}$, we notice, for instance by observing a, that relation R' contains one more property than R ("$a < d$") and that this depends only on the fact that the set A' to which it refers contains one element more than $A(d)$, i.e. it depends

[19] Yet, he specified [1914, 62] that he did not share Dedekind's concept of "the smallest chain", because it did not seem evident to him. Furthermore, he disapproved of the fact that Dedekind presupposed logic as a noncontradictory construction. Instead, according to König, one has to prove (as he will do) that logic is noncontradictory.

[20] "Wir haben es nicht mit einer Schranße unseres Denßvermögens, wohl aber mit einer Schranße unseres Anshaungsvermögens zu tun."

[21] König added [1914, 56] that order relations have a meaning only when they have to do with things which are different from each other.

on the things that it links[22]. As we shall see later, this characteristic of order relations will be useful in solving a certain kind of paradox. Order relations have to do with representations because an order relation has to be applied to representations in order to allow the process of counting, that consists of putting things in bijective correspondence with representations[23].

Its axiomatic form (i.e., the axiomatic form of complete induction) is the following:

$$1 \,\mathrm{qual}\, I$$
$$[x \,\mathrm{qual}\, I] \rightarrow [f x \,\mathrm{qual}\, I]$$

where "I" means "inductive numbers". This form is acceptable because it expresses something that is an act of thought, as we have been seeing till now. In this sense, König noticed[24] that "logical deduction as such is, in the case of complete induction, an object of intuition" [1914, 159], where the "logical deduction" is what we have called "act of thought". Since this form is acceptable, it grants that for all numbers k "$k \,\mathrm{qual}\, I$" belongs to the *Denkbereich*. This axiomatic form can be generalised, by interpreting "I" as any other quality at will[25]. This means that the axiomatic form above expresses what is commonly called "induction procedure", which is acceptable inside König's system. Namely, he proved that such form is noncontradictory stressing that if one starts by assigning "$z \,\mathrm{qual}\, I$" to the P-forms (true forms), then the result of the application of induction schema is again a P-form.

On this purpose, we recall here Poincarè's analogous viewpoint [1902, 23–24]: "[le raisonnement par récurrence] … *Pourquoi donc ce jugement s'impose-t-il à nous avec une irrésistible évidence? C'est qu'il n'est que l'affirmation de la puissance de l'esprit qui se sait capable de concevoir la répétition indéfinie d'un même acte dès que cet acte est une fois possible. L'esprit a de cette puissance une intuition directe et l'expérience ne peut être pour lui qu'une occasion de s'en servir et par là d'en prendre conscience*".

§7. Solving paradoxes. Among the forms of the axiomatised set-theory the following is particularly interesting:

$$[M \sim\sim v] = [M \sim\sim v'].$$

[22]Another important empirical relation is that of "copula" (a is b), i.e. "equality". According to König [1914, 24], it is empirical because it cannot be independent of the things that it links, and links cannot be caught by intuition without the linked things.

[23]Concerning this, König observed [1914, 58] that the concept of "amount" derives from that of counting and not viceversa.

[24]"Die logische Deduktion als solche ist in der vollständiger Induktion Gegenstand der Anschauung".

[25]König observed [1914, 159] that it can also be generalised in another —less interesting— way, by using another symbol instead of "1", and starting, so, with another "thing", different from "1".

It can be added as an axiom if M is an "N^*-form" (i.e., if M is neither true nor false) and remains such under any substitution. That is to say, in this case (and only in this case), it entails no contradiction, because it makes no difference whether M belongs to the *Denkbereich* or not, insofar as neither $M \sim\sim v$ nor $M \sim\sim v'$ belongs to it. Hence, adding it as an axiom, automatically means requiring that "neither $e \sim\sim v$ nor $e \sim\sim v'$ belong to the *Denkbereich*" (which König called a "sophistic" *Denkbereich* [1914, 151–153]) because this is the condition for the axiom not to cause contradiction, and not causing contradiction is a condition for a form to be accepted as an axiom[26]. König added [1914, 83] that also $[M \sim\sim v] \leftrightarrow [M \sim\sim v']$ can indifferently be added (for the same reason as "="), where $A \leftrightarrow B$, called relation of equipollence, means "$A \rightarrow B$ & $B \rightarrow A$".

The interest of this axiomatic form arose from König's use of it in order to solve both the Russell and the liar paradoxes. We recall here that the Russell paradox appears when a set is defined as the set of sets that do not belong to themselves. If one asks whether or not the defined set belongs to itself, then the paradoxal result is obtained: either the set belongs to itself or not, but if the set belongs to itself, then (by definition) it cannot belong to itself; if it does not belong to itself, then (by definition) it belongs to itself. The traditional expression of the liar paradox is the following: if the Cretan Epimenides says "the Cretans are liars", is he telling a lie? Either he is telling a lie or not, but if he is telling a lie, then he is telling the truth; if he is telling the truth, then he is telling a lie.

These paradoxes are based on two facts:

1) they have to do with definitions, where the falsity of a certain form implies its truth and viceversa;
2) the paradoxes take place if the law of bivalence holds.

As for Russell's paradox, he added [1914, 153] as a definition of the set concerned the following axiomatic form:

$$[(x \,\mathrm{elem}_\alpha\, x) \sim\sim v'] \leftrightarrow [x \,\mathrm{elem}_\alpha\, R].$$

Then, he performed the substitution of x by R, obtaining:

$$[(R \,\mathrm{elem}_\alpha\, R) \sim\sim v'] \leftrightarrow [R \,\mathrm{elem}_\alpha\, R].$$

The last member of this equipollence is clearly equivalent to $(R \,\mathrm{elem}_\alpha\, R) \sim\sim v$, hence the whole equipollence can be written as follows:

$$[(R \,\mathrm{elem}_\alpha\, R) \sim\sim v'] \leftrightarrow [(R \,\mathrm{elem}_\alpha\, R) \sim\sim v].$$

This is a form like $[M \sim\sim v] \leftrightarrow [M \sim\sim v']$ above. Its presence among the axioms automatically means that both

$$\text{“}(R \,\mathrm{elem}_\alpha\, R) \sim\sim v'\text{”} \quad \text{and} \quad \text{“}(R \,\mathrm{elem}_\alpha\, R) \sim\sim v\text{”}$$

[26]We recall here what we wrote in Section 2, that there can be *Denkbereiche* about which one cannot establish for any *Erlebnis* whether it belongs to the *Denkbereich* or not.

do not belong to the (sophistic) *Denkbereich*, i.e. that the law of bivalence does not hold for $R \,\mathrm{elem}_\alpha\, R$. As each of them would entail the other, the fact that neither of them can be established solves the problem at its very origin.

The same treatment is valid for the liar paradox. In this case, König added [1914, 154] the following axiomatic form as a definition of the set concerned:

$$[e \sim\sim v] = [e \sim\sim v']$$

where "e" means "Epimenides is telling a lie".

This form is like $[M \sim\sim v] = [M \sim\sim v']$ above, therefore its presence among the axioms automatically means that both $e \sim\sim v$ and $e \sim\sim v'$ do not belong to the (sophistic) *Denkbereich*, i.e. that the law of bivalence does not hold for e. Hence, the question "is Epimenides telling a lie?" has no answer in that *Denkbereich* (because it should be either $e \sim\sim v$ or $e \sim\sim v'$). As its answer would be the cause of the paradox (because "false" entailed "true" and viceversa), the absence of the answer from the *Denkbereich* solves the problem.

It is useful to stress that König rejected the law of bivalence (although he did not call it thus), but was conscious that this rejection was very different from affirming its negation. He explicitly stated [1914, 152] that among the axiomatic forms of an axiomatised theory there cannot be, for instance, "neither $e \sim\sim v$ nor $e \sim\sim v'$ belonging to the *Denkbereich*", because it would be a contradiction (like "$\neg A$ & $\neg\neg A$").

König also dealt with the Berry-Dixon paradox and the largest ordinal paradox.

Berry-Dixon's paradox consisted in showing that the name of a certain number contains and does not contain a certain number of words at the same time. The number would be the smallest integer whose name contained at least n words: if one takes, for instance, $n = 100$, then "the smallest integer whose name contains at least 100 words" is the name, containing less then 100 words, for an integer whose name should contain at least 100 words. According to König[27] [1914, 213], "the smallest integer whose name contains at least 100 words" is not "*a name for that number, but for the definition of that number. A number does not correspond to this name, but rather a rule for distinguishing this number for all other things*".

Finally, for the paradox of the largest ordinal number, König solved it as follows: when we add an element to an ordered set, we change the order in it, according to the fact that order is an empirical relation; therefore, it is impossible to speak about the set of all ordinals, because this should contain only one kind of order (as it is one set), while it contains infinitely many

[27] "[...] ist kein Schriftname jener Zahl, sondern Schriftname einer Definition dieser Zahl. Ihm zugeordnet ist nicht die Zahl, sondern eine Regel, nach der diese Zahl von jedem andern Dinge unterschieden werden kann."

kinds of orders (corresponding to each ordered set which gives place to each ordinal number). Hence, no paradox can take place.

It is worth stressing that König did not mention the paradox bearing his name. He had exposed it in his 1906 article. The paradox regarded the finite definability of real numbers: the set of all finitely definable numbers is denumerable and can be written in a matrix according to Cantor:

$$\begin{array}{cccc} a_{11} & a_{12} & a_{13} & \dots \\ a_{21} & a_{22} & a_{23} & \dots \\ a_{31} & a_{32} & a_{33} & \dots \\ & \dots & & \end{array}$$

Yet, again by Cantor's diagonal procedure, König formed a real number which differed from each row and column of the matrix because he added an integer d to each a_{nn}. The real number was finitely defined but it did not belong to the matrix, i.e. to the set of all finitely definable real numbers. Hence, the paradox was very similar to that of Berry. Therefore, in [1914], König also considered his paradox as apparent and did not mention it expressly. He only clarified in a footnote [1914, 214] that considering the smallest integer whose name contains at least 100 words not as a name for a number, but as a name for the definition of a number, was his final opinion on the subject that cancelled all that he had written on it before.

§8. Innovations in König's work. Looking at König's work, we can now draw some conclusions. At a first sight, König's psychologism and also his references impose a comparison with algebraists of logic. As we already recalled, he did not mention any of them specifically, but we can make a general comparison with them. There is surely a different way of theorizing sets: for König they are not the result of an act of conceiving[28] but the result of an act of attribution. Furthermore, he offers a subtler analysis of the concept "set": he includes naming as a particular kind of forming sets and he distinguishes between a more abstract level, when only the attribution is considered, and a more concrete level, where also the existing things (towards which the attribution is directed) are considered. The reference to the evidence is the same[29] , yet König affirms [1914, 105f.] that the belief in the trustability of evidence cannot be considered a postulate[30] because we would postulate a property of the *Denkbereich* as a further intuition while the *Denkbereich* is a definitive object of intuition. Therefore, we only have this belief and that is all. Finally, there is the suggestion by König himself that his main improvement over logical algebraists consists in a change of way of thinking. What does this exactly mean? He does not use psychologism,

[28] See Boole [1847, Ch. 3 §6].

[29] See Schröder [1890, 7].

[30] As Schröder [1890, 7] does by referring to Sigwart [1873,15]. König quoted this latter explicitely.

i.e. the reference to the reality of conscience, simply to list the laws of our thought. Namely, each time he introduces a new (form of) set or when he writes down a formal logical system he puts the requirement that they have to be proved to be consistent. This is done —so to say— in the name of the *Grundnorm* as in any case we describe mental states. On this purpose, it is interesting to stress that König, in the introduction to his book, saw [1914, iv] an analogy between the intuition of numbers —to which Hilbert planned to reduce the consistency of geometry— and the intuition of the states of consciousness —to which he planned to reduce the consistency of logic[31]. Therefore, we can say that the new use that König makes of psychologism can be seen as an "improvement" of both logical algebraists and Hilbert. With respect to logical algebraists, he does not limit himself to a formalization but also requires the consistency of logical forms/systems. As for Hilbert, we can say that König points out how the last ground of his theorization lies in consciousness. Consistency is for Hilbert the condition for the logical existence, where this means nothing else that "thinkability". König specifies that consistency is the condition for thinkability because stating the thinkability of something amounts to positing the possibility of a mental state and any mental state must obey the *Grundnorm*. Hilbert himself had to refer to mental states/intuition for grounding his theorization, but he did so not for what concerned thinkability/consistency but for what concerned the basic elements of mathematics (which he later will use for proving the consistency of the rest of mathematics). For Hilbert, thought was something different from mental states/intuition. Already in 1905 Hilbert referred to *mental things*, called through signs, that, suitably combined, should be the basis of mathematics[32]. The statement of their existence was called[33] by him "the axiom of thought" (or "axiom of the existence of an intelligence") and was accompanied by the additional note "it is the a priori of philosophers", in the sense of *pre*-condition for *thought*. That is, mental things come *before* thought. Later, after Brouwer's appearance on the foundational scene, signs themselves, described[34] as "certain extralogical discrete objects, that come intuitively as direct *Erlebnis* before any thought", will be considered the basis of mathematics, that could be shared also by intuitionists as something evident, coming from intuition, described as a third source of knowledge *in addition to* experience and *thought*[35]. Hence Hilbert went on with his

[31] "Aehnlich wie Hilbert die Widerspruchslosigkeit seiner 'Geometrien' durch Zurückführung auf das Anschauungsgebiet der Zahlen erhärtet, wird die Widerspruchslosigkeit des logischen Denkens durch Reduktion auf jenes eben erwähnte Anschauungsgebiet 'evident'".

[32] See [1905a, 177] and [1905b, 219].

[33] [1905a, 177].

[34] "Gewisse ausserlogische diskrete Objekte, die anschaulich als unmittelbares Erlebnis vor allem Denken da sind" [1922, 162–163].

[35] See [1926, 171].

opinion and terminology and considered "thought" something equivalent to "logical" at its turn equivalent to "noncontradictory" (without further explanation), and mental/intuitive as something that comes before all of this as an isle of surety on which to prove the consistency (thinkability, logical existence) of formal systems.

On the purpose of Hilbert, we have to recall that John von Neumann in his "Zur Hilbertschen Beweistheorie" stressed [1927, 22] that Hilbert had got from König the idea of proving the noncontradictoriness of mathematics through what von Neumann himself called an "evaluation" (*Wertung*). On his side, Paul Bernays [1935, 200] criticised this affirmation from an historical viewpoint and stated that between the two authors there had been only a similarity in the content of the idea of *Wertung*: namely, both of them split formulas (after some suitable operations on them) into two classes (true/false) and then prove that axioms belong to the class of true formulas and that logical rules let us pass from true formulas only to true formulas. Still, after that Hilbert concludes the noncontradictoriness of his basic system (without both quantifiers and ε/τ-operators) by observing that if $0 \neq 0$ could be derived from its axioms through its rules it should be true, while it is not; on his side, after splitting formulas into the two classes, König concludes directly the noncontradictoriness of his logical system as all theorems belong to the same class (true). In spite of this (partial) similarity, Bernays stated, there was not a real influence of König on Hilbert.

In order to point out some further aspects of König's originality, we have now to consider his looking for a "concordance" between Poincaré and Cantor to see which interesting innovations it brings with it. We need to establish more in detail what this concordance consisted of. It is obvious that this concordance can only partially satisfy Poincaré's requirements on mathematics as Cantorians were strongly criticized by Poincaré himself. In fact, we can individuate the very core of this concordance in the notion of noncontradictoriness. König used the search for noncontradictoriness —which was in line with Poincaré's idea that noncontradictoriness grants mathematical existence [1905, 819], [1906, 296–299]— as a method for securing Cantor's set theory against paradoxes that Poincaré had judged and solved in a different way. König expressed this fact by considering the Berry-Dixon paradox. He recalled the two observations by Poincaré[36] about this paradox (under the form given by Richard) and replied to them. On the one hand, König underlined that impredicative definitions in themselves are not always contradictory. As an example, he gave the following [1914, 213]: "The smallest integer whose name in German contains more than one hundred letters is N (where, for N, the name for that number is still to be established)". On the other hand, he specified that in the case of the Berry-Dixon paradox, although the partition of all names of integers into classes according to the

[36][1906, 307], [1909, 461–465].

number of words that define each of them cannot be considered as completed in advance, it is however clear how the paradox works (what "a number defined by 100 words" means) and therefore this is not a suitable criticism (and hence a solution) of the paradox. König's different interpretation of the causes of paradoxes with respect to Poincaré's implied that he did not consider them as a ground for refusing the actual infinite, and, hence, he could restrict his attention to simply avoiding noncontradictoriness in each single definition of sets. However, even his notion of noncontradictoriness has something different from that proposed by Poincaré. The difference concerns the relationship between mathematics and logic. As we have seen before, König highly estimated logical systems. He believed that both mathematics and pure logic have to reveal the content of mental states starting from what our intuitions tell us about them and using logical systems for discovering further aspects of them. Noncontradictoriness is the first step for later using logical tools in order to obtain mathematical results. On his side, Poincaré was convinced [1905, 819–820] that noncontradictoriness is the condition for mathematical existence and is proved inside mathematics, but then he added [1894,1908] that mathematics is something creative and, therefore, discovering theorems is an activity that does not follow any rule given in advance. It is helped by intuition for establishing which principles have to be used to reach some results. Hence, for Poincaré noncontradictoriness is not the first step in order to use logical tools for obtaining mathematical results. Still, this role attributed by Poincaré to intuition shows a further aspect of his thought shared by König. In fact, for Poincaré [1906, 307] intuition grants that both induction and the axiom of choice are synthetical a priori principles that "work" in mathematics. For this reason, Poincaré had accepted both principles. Also König grounded them on intuition [1914, 160]. As for the fact that those principles "work" in mathematics, we can hypothesize an allusion to it when König stated [1914, 172] that showing that the axiom of choice was based on intuition should satisfy Borel's requirement for mathematical entities: "*Il n'y a pour lui [le mathématicien] qu'une distinction utile: certaines définitions permettent de connaître parfaitement l'être mathématique dont on parle, de calculer sur lui sans aucune ambiguïté ou confusion possible, d'autres ne le permettent pas*" and, hence, eliminate his doubts about the axiom. He added that the same requirement had been put by Poincaré "in another direction" (*in anderer Richtung*). König did not specify this assertion any more: we can interpret it in the sense that he alluded to the fact that, for both Borel and Poincaré, definitions are admissible when they "work" in mathematics, i.e. when they allow precise calculations; still this requiment had led Borel to refuse the axiom of choice, while the same requirement had led Poincaré in the other direction, that is, to the acceptance of the axiom. The partial acceptance of Poincaré's thought in order to conciliate it with Cantor's set theory, therefore, is an original aspect of König's book.

Finally, we have to consider his introduction of a third truth-value. In the early history of many.valued logics, the founding fathers are usually considered Hugh MacColl, Charles Sanders Peirce and Nikolai Vasil'ev. For the present article the figure of highest interest is Hugh MacColl as Vasil'ev was publishing just in 1910–1912 his papers in Russian as reports of the Kazan University while, as for Peirce, "there is now conclusive evidence that by early 1909 he had —with considerable elaborateness— extended the truth-table method to 3-valued logic" [Rescher 1969, 5]: that is, there is no reasonable ground for believing that König was able to read them at his time. On the contrary, Hugh MacColl already in the part of 1897 of his article "Symbolic Reasoning", published in *Mind*, and then, more extensively, in his 1906 book *Symbolic Logic and its Applications*, had proposed a system of propositional logic that he defined as "logic of three dimensions" (=truth-values), according to which propositions can take on the modal values of "certainty", "impossibility" and "variability". A certain proposition is one which is always and necessarily true, an impossible proposition is one which is always and necessarily false, and a variable proposition is one which is sometimes true and sometimes false. He developed his system as an algebra of logic along the lines of previous systems, but based on three rather than two truth-values, and applied it to the calculus of probability. König did not mention MacColl. In any case, the work of this author differs from his own not only for the domain of applicability but also for the meaning of the third "truth-value": as it is a modal value, it means *sometimes true/sometimes false* (an the alleged example was: "$x = 2$"), while the third possibility for König was the admission of the impossibility of establishing whether the proposition was true or false. Therefore, we can insert Julius König among the founding fathers of many-valued logics.

This is a further aspect of originality to be added to the many innovations that we have already seen throughout this paper and that it is useful to recall here: a notion of metatheory (theory at a higher level of another), the non-acceptance of the law of excluded middle among the axiomatized logical laws and the solution of paradoxes based on the refusal of the law of bivalence.

REFERENCES

BERNAYS P. (1935), *Hilberts Untersuchungen über die Grundlagen der Arithmetik*, ***D. Hilbert Gesammelte Abhandlungen***, vol. 3, Springer, pp. 196–216.

BOOLE G. (1847), ***Mathematical Analysis of logic***, Macmillan, Barclay and Macmillan / George Bell, Cambridge / London.

BROUWER L.E.J. (1907), ***Over de grondslagen der wiskunde***, Maas and van Suchtelen, Amsterdam, (English translation, *On the Foundations of Mathematics*, **L.E.J. Brouwer Collected Works** (A. Heyting editor), vol. 1 (1975), North-Holland, Amsterdam, pp. 11–101).

——— (1908), *De onbetrouwbaarheid der logische principes*, ***Tijdschrift voor Wijsbegeerte***, vol. 2, pp. 152–158, (English translation, *The Unreliability of Logical Principles*, **L.E.J.**

Brouwer Collected Works (A. Heyting editor), vol. 1 (1975), North-Holland, Amsterdam, pp. 107–111).

——— (1912), ***Intuitionisme en formalisme; inaugurale rede***, Clausen, Amsterdam, (English translation, *Intuition and Formalism*, **L.E.J. Brouwer Collected Works** (A. Heyting editor), vol. 1 (1975), North-Holland, Amsterdam, pp. 123–138).

DEDEKIND R. (1888), ***Was sind und was sollen die Zahlen?***, Vieweg, Wiesbaden.

DESCARTES R. (1963), ***Œuvres et lettres***, Pleiade, Paris.

FREGE G. (1879), ***Begriffsschrift***, Nebert, Halle.

——— (1893), ***Grundgesetze der Arithmetik***, Pohle, Jena.

GRASSMANN H. (1861), ***Lehrbuch der Arithmetik für höhere Lehranstalten***, Th. Chr. Fr. Enslin., Berlin.

HALLETT M. (1984), ***Cantorian Set Theory and Limitation of Size***, Oxford University Press, Oxford.

HILBERT D. (1905a), ***Über die Grundlagen der Logik und der Arithmetik, Verhandlungen des dritten internationalen Mathematiker-Kongresses in Heidelberg vom 8. bis 17. August 1904***, Teubner, Leipzig, (English translation, *On the Foundations of Logic and Arithmetic,* ***From Frege to Gödel: a Source Book in Mathematical Logic*** (J. van Heijenoort, editor), 1976, Harvard University Press, Cambridge, pp. 130–138).

——— (1905b), ***Logische Principien des mathematischen Denkens***, Vorlesungen ausgearbeitet von Ernst Hellinger.

——— (1922), *Neubegründung der Mathematik. Erste Mitteilung*, ***Abhandlungen aus dem mathematischen Seminar der Hamburgischen Universität***, vol. 1, pp. 157–177.

——— (1926), *Über das Unendliche*, ***Mathematische Annalen***, vol. 95, pp. 161–190, (English translation, *On the Infinite*, ***From Frege to Gödel: a Source Book in Mathematical Logic*** (J. van Heijenoort, editor), 1976, Harvard University Press, Cambridge, pp. 369–392).

(1972), ***Historisches Wörterbuch der Philosophie***, vol. Bd II, Schwabe, Basel.

KNEALE W. and KNEALE M. (1962), ***The Development of Logic***, Oxford University Press, Oxford.

KÖNIG J. (1905a), *Zum Kontinuum-Problem*, ***Mathematische Annalen***, vol. 60, pp. 177–180.

——— (1905b), *Über die Grundlagen der Mengenlehre und das Kontinuumproblem*, ***Mathematische Annalen***, vol. 61, pp. 156–160.

——— (1906), *Sur la théorie des ensembles*, ***Comptes rendus de l'Académie des Sciences de Paris***, pp. 110–112.

——— (1907), *Über die Grundlagen der Mengenlehre und das Kontinuumproblem*, ***Mathematische Annalen***, vol. 63, pp. 217–221.

——— (1914), ***Neue Grundlagen der Logik, Arithmetik und Mengenlehre***, Von Veit, Leipzig.

MACCOLL H. (1897), *Symbolical Reasoning*, ***Mind***, vol. 6, pp. 493–510.

——— (1906), ***Symbolic logic and Its Applications***, Longmans, Green & Co., London, New York, Bombay.

MOORE G.H. (1978), *The Origins of Zermelo's Axiomatization of Set Theory*, ***Journal of Philosophical Logic***, vol. 7, pp. 307–329.

——— (1982), ***Zermelo's Axiom of Choice: its Origins, Development, and Influence***, Springer, New York.

J. VON NEUMANN (1927), *Zur Hilbertschen Beweistheorie*, ***Mathematische Zeitschrift***, vol. 26, pp. 1–46.

PEANO G. (1889), ***Arithmetices principia nova methodo exposita***, Bocca & Clausen, Torino.

PECKHAUS V. (1990), ***Hilbertprogramm und kritische Philosophie***, Vandenhoeck & Ruprecht, Göttingen.

——— (1994), *Logic in Transition: the Logical Calculi of Hilbert (1905) and Zermelo (1908)*, ***Logic and Philosophy of Science in Uppsala*** (D. Prawitz and D. Westerstähl, editors), Kluwer, Amsterdam, pp. 311–323.

——— (1996), *The influence of Hermann Günther Grassmann and Robert Grassmann on Ernst Schröder's Algebra of Logic*, ***Hermann Günther Grassmann (1809–1877): Visionary Mathematician, Scientist and Neohumanist Scholar*** (G. Schubring, editor), Kluwer, Dordrecht, Boston, London, pp. 217–227.

Poincaré H. (1894), *Sur la nature du raisonnement mathématique*, ***Revue de métaphysique et de morale***, vol. 2, pp. 371–384.

——— (1902), ***La Science et l'Hypothèse.***, Bibliothèque de philosophie scientifique, Paris.

——— (1905), *Les mathématiques et la logique*, ***Revue de métaphysique et de morale***, vol. 13, pp. 815–835.

——— (1906), *Les mathématiques et la logique*, ***Revue de métaphysique et de morale***, vol. 14, pp. 294–317.

——— (1908), *L'invention mathématique*, ***L'enseignement mathématique***, vol. 10, pp. 357–371.

——— (1909), *La logique de l'infini*, ***Revue de métaphysique et de morale***, vol. 17, pp. 461–482.

Post E.L. (1921), *Introduction to a General Theory of Elementary Propositions*, ***American Journal of Mathematics***, vol. 43, pp. 163–185.

Rescher N. (1969), ***Many-valued Logic***, McGraw-Hill, New York.

Russell B. and Whitehead A.N. (1910), ***Principia Mathematica I***, Cambridge University Press, Cambridge.

——— (1911), ***Principia Mathematica II***, Cambridge University Press, Cambridge.

——— (1912), ***Principia Mathematica III***, Cambridge University Press, Cambridge.

Schröder E. (1890), ***Vorlesungen über die Algebra der Logik***, vol. 1, Teubner, Leipzig.

Sigwart C. (1873), ***Logik***, 3rd ed., Taupp, Tübingen, (1904), (This was the edition to which König referred).

Szénássy B. (1992), ***History of Mathematics in Hungary until the 20th century***, Springer, Berlin.

Wang H. (1957), *The Axiomatization of Arithmetic*, ***Journal for Symbolic Logic***, vol. 22, pp. 145–158.

Wittgenstein L. (1921), *Logisch-philosophische Abhandlung*, ***Annalen der Naturphilosophie***, vol. 14, pp. 185–262.

Zermelo E. (1904), *Beweis, dass jede Menge wohlgeordnet werden kann*, ***Mathematische Zeitschrift***, vol. 59, pp. 514–516.

——— (1908), *Untersuchungen über die Grundlagen der Mengenlehre I*, ***Mathematische Annalen***, vol. 65, pp. 261–281.

DIPARTIMENTO DI FILOSOFIA
UNIVERSITÀ DEGLI STUDI
V. FESTA DEL PERDONO 7, 20122 MILANO, ITALY
E-mail: Miriam.Franchella@unimi.it