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DOMAINS FOR COMPUTATION IN MATHEMATICS, PHYSICS AND EXACT REAL ARITHMETIC

ABBAS EDALAT

Abstract. We present a survey of the recent applications of continuous domains for providing simple computational models for classical spaces in mathematics including the real line, countably based locally compact spaces, complete separable metric spaces, separable Banach spaces and spaces of probability distributions. It is shown how these models have a logical and effective presentation and how they are used to give a computational framework in several areas in mathematics and physics. These include fractal geometry, where new results on existence and uniqueness of attractors and invariant distributions have been obtained, measure and integration theory, where a generalization of the Riemann theory of integration has been developed, and real arithmetic, where a feasible setting for exact computer arithmetic has been formulated. We give a number of algorithms for computation in the theory of iterated function systems with applications in statistical physics and in period doubling route to chaos; we also show how efficient algorithms have been obtained for computing elementary functions in exact real arithmetic.

§1. Introduction. Domain theory was introduced by Scott [105] in 1970 as a mathematical theory of computation in the semantics of programming languages. Some earlier ideas of the subject had appeared in the work of Lacombe [85] in recursion theory. A number of fundamental contributions to the theory were also made independently by Ershov in the context of partial computable functionals of finite type [49].

A domain is a structure for modeling a computational process or a data type with incompletely specified elements. It is a partially ordered set with the partial order corresponding to some notion of *information*. In order to model computational processes as a sequence of finite steps, a domain is equipped with a notion of *finite elements* and a notion of *limits* provided by a least upper bound operation. A simple example is given by the set of finite and infinite sequences $\Sigma^\infty = \Sigma^* \cup \Sigma^\omega$ over a fixed alphabet Σ. The elements of Σ^∞ can represent partial or total output of a computation. They are partially ordered with pre-fix ordering $\sqsubseteq$, that is to say, for $x, y \in \Sigma^\infty$ we have $x \sqsubseteq y$ if the sequence x is an initial segment of the sequence y, i.e., if x gives less information than y. Every increasing chain of sequences in Σ^∞ has a least upper bound, namely the least sequence with respect to the prefix ordering which has each sequence in the chain as an initial segment.

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Furthermore, every sequence in Σ^∞ is the least upper bound of a chain of finite sequences which give finite approximations to that sequence.

The Scott model can be described as follows. Given a computation based on an algorithm, the set of input and the set of output each forms a domain. The program which carries out the computation is represented as a function between these two domains. Every new step in the computation results in an element in the domain of output which provides more information and a better approximation to the ultimate result. This final outcome may be attained in a finite number of steps or may only be captured by the limit of an infinite sequence of steps of computation. Scott's thesis is that any computable function is continuous: It preserves the information order (so that more information as input gives more information as output) and the limits of infinite computations in the domain (so that the total information obtainable as output from an infinite sequence of input elements with refining information is the sum total of all the information obtained from each input element).

Programs with the same domains of input and output are in turn pointwise ordered to yield a domain of functions or a function space. Thereby, one is able to model higher order functions or higher order programs which can take a program as input. Any continuous function on a domain with a least element has a least fixed point as in Tarski's theorem. This implies that a recursive program can be captured as the fixed point of a higher order function which is defined, by the corresponding recursion, on the domain of all programs of the given type.

There are a number of basic categories of domains according to various additional properties that they satisfy [79, 2]. Algebraic domains are characterized by a subset of so-called finite or compact elements representing computational results which can be obtained in a finite number of steps. The finite elements form a *basis* of the domain; every element of the domain is the limit of the basis elements approximating it. An algebraic domain with a countable basis, called an ω-algebraic domain, can be effectively presented to make the theory constructive and to define the notion of computable element and computable function. Scott domains form a particularly simple class of ω-algebraic domains in which every bounded subset has a least upper bound.

The basis elements can also be regarded as a set of logical propositions which characterize any element of the domain. This was first noted by Scott [108] for Scott domains and was later generalized for other classes of domains: An algebraic domain has a simple presentation in terms of an *information system*, a logical structure on the basis elements which gives a prescription how to construct the elements as the theories of the corresponding logic. The logic underlying an algebraic domain is that of the observable properties of the computation process. This "logic of observation" is closely linked to the Scott topology of domains as noted by Smyth [112] and elaborated by Vickers [118]. A Scott open subset can be viewed as a proposition about, or a property of, a program. A comprehensive analysis of the underlying logic for the cartesian closed category of the so-called bi-finite domains was developed by Abramsky [1]. Various other categories of domains have also been studied in logical form [45, 119, 72].

Several cartesian closed categories of algebraic domains, including the so-called Scott domains, have been employed in the semantics of computation. They are used to obtain a non-trivial model of the untyped λ-calculus [6] based on a domain isomorphic to its own function space [106, 107]. They have also provided a denotational semantics for PCF [95] (Programming Language for Computable Functions), essentially a typed λ-calculus with ground types for natural numbers and Boolean values plus constants for basic operations on these types; PCF can be considered as the theoretical model for functional programming languages. Domain theory has developed extensively in the past three decades and is now a major paradigm in the semantics of programming languages. For a basic introduction to its theory and applications, see [59, 74, 96, 64, 2, 114].

Algebraic domains have also been used to represent classical spaces in mathematics in an effective framework. Weihrauch and Schreiber [124] constructed an embedding of a Polish space (a topologically complete separable metrizable space) into an algebraic domain. Stoltenberg-Hansen and Tucker have shown how to represent complete local rings [115] and topological algebras, in a general setting, including locally compact Hausdorff spaces and the real line, by algebraic domains [116]. Jens Blanck [19] has more recently shown how to embed a complete metric space into an algebraic domain.

In recent years, a new direction for application of domains in computation on classical spaces in mathematics has emerged. Continuous domains are generalizations of algebraic domains and share many of their basic properties; in fact every continuous domain is a retract of an algebraic domain, and one can move from a representation by an algebraic domain to one by a continuous domain and vice versa. However, continuous domains are the natural setting for continuous mathematics since the representations they provide are far more direct and straightforward than those by algebraic domains. In fact, Scott in [105] had suggested that the continuous domain of the compact intervals of the real line can be used as a data type for real numbers.

In [35], the author presented the notion of a domain-theoretic computational model for a countably based locally compact Hausdorff space: it was shown that the continuous domain obtained by taking the non-empty compact subsets of the space, ordered by reverse inclusion, provides a simple and effective model for computation on this classical space which is embedded onto the subspace of the maximal elements of the domain. This domain, called the upper space, is equipped with a countable basis of non-empty compact subsets which can be enumerated to give an effective structure for the domain. An element of the locally compact space identified as a singleton element can be obtained as the intersection of a shrinking nested sequence of the basis elements. A computable element of the domain, in particular a computable element of the locally compact space, is the intersection of an effective shrinking sequence of basis elements.

This led to a new framework for iterated function systems, measure theory and integration with applications in fractal geometry and statistical physics. More specifically, a generalization of the Riemann theory of integration has been developed which retains the computational and constructive features of the ordinary Riemann integral: it has provided a new technique for computation of integrals. Later, computational models for complete separable metric spaces were also constructed. Furthermore, similar computational models have provided a framework for exact real number computation leading to efficient algorithms in infinite precision computer arithmetic.

In this article we give an outline of the new applications of continuous domains in mathematics, physics and real number computation based on various computational models in these areas. The aim is to give a precise account of the results obtained which would also be self-contained. We will only point out some of the basic results in the subject; for proofs, other basic results and various generalisations the reader is referred to the relevant papers.

We start in Section 2 by defining the basic tools in domain theory which we will use to construct our computational models and present them in a logical form and in an effective framework. We then give, in Section 3, some basic examples of these models for real numbers, locally compact spaces and complete metric spaces. In Section 4, dynamical systems on domains are studied and the domain-theoretic models are used to obtain various new results in the theory of iterated function systems in fractal geometry. In Section 5, we construct a computational model for classical measure theory on locally compact second countable Hausdorff spaces. In Section 6, the new model for measures is used to give a computational generalisation of the Riemann theory of integration. Section 7 presents various new results in the theory of iterated function systems with probabilities. In Section 8, two applications in computing various quantities in statistical physics and chaos theory are presented. Finally, in Section 9, we outline the domain-theoretic approach in exact real number computation.

§2. Directed complete and continuous posets. In this section, we present the basic notions of domain theory which we need in this exposition and explain how continuous domains can be presented in logical form and how they can be effectively presented. We use the following conventions in this paper. For any map $f : D \to E$, any point $x \in D$, any subset $A \subseteq D$ and any subset $B \subseteq E$, we denote, whenever more convenient, the image of x by fx instead of $f(x)$, the forward image of A by fA instead of $f(A)$ and the pre-image of B by $f^{-1}B$ instead of $f^{-1}(B)$. The set of all finite subsets of the set S is denoted by $\mathcal{P}_f(S)$ and the lattice of open sets of a topological space X by $\Omega(X)$.

2.1. Basic definitions. A non-empty subset $A \subseteq P$ of a poset $(P, \sqsubseteq)$ is *directed* if for any pair of elements $x, y \in A$ there is an upper bound $z \in A$ with $x, y \sqsubseteq z$. An increasing chain is the simplest example of a directed set. If we think of the poset P as the set of inputs or outputs, then a directed set A corresponds to a consistent set of inputs or outputs of a given program: for any two elements in A there exists an element which refines the information of both. We therefore require that in a domain of computation the total information in a directed subset should be represented by an element of the domain, in other words a domain should contain the least upper bounds of directed subsets. A *directed complete partial order* (*dcpo*) is a partial order in which every directed subset A has a least upper bound (lub), denoted by $\bigsqcup A$.

An open set $O \subseteq P$ of the *Scott* topology of P is a set which is upward closed (i.e., $x \in O$ & $x \sqsubseteq y \;\Rightarrow\; y \in O$) and is inaccessible by lubs of directed sets (i.e., if A is directed with a lub, then $\bigsqcup A \in O \;\Rightarrow\; \exists x \in A.\, x \in O$). Dually, a closed set $C \subseteq P$ of the Scott topology of a poset P is a set which is downward-closed (i.e., $x \in C$ & $y \sqsubseteq x \;\Rightarrow\; y \in C$) and is closed under the lubs of directed subsets (i.e., for any directed subset $A \subseteq C$ with lub we have $\bigsqcup A \in C$). The Scott topology of any poset is T_0. The *Lawson* topology is a refinement of the Scott topology in which subsets of the form $\uparrow d = \{x \in P | d \sqsubseteq x\}$ are also closed. Unless otherwise stated, the topology of a poset in these notes is always assumed to be the Scott topology.

The *function space* $D \to E$ of two dcpo's D and E is the set of continuous functions $f : D \to E$ with the pointwise ordering: $f \sqsubseteq g$ if $\forall x \in D.\, f(x) \sqsubseteq g(x)$. Then $D \to E$ is a dcpo where the lub $\bigsqcup_{i \in I} f_i$ of a directed subset in $D \to E$ is given by $(\bigsqcup_{i\in I} f_i)(x) = \bigsqcup_{i \in I}(f_i(x))$. It can be shown that a function $f : D \to E$ from a dcpo D to another one E is continuous with respect to the Scott topology iff it is *monotone*, i.e., $x \sqsubseteq y \Rightarrow f(x) \sqsubseteq f(y)$, and preserves lubs of directed sets, i.e., $\bigsqcup_{i\in I} f(x_i) = f(\bigsqcup_{i \in I} x_i)$, where $\{x_i \mid i \in I\}$ is any directed subset of D. From this one obtains a Tarski-like fixed point theorem: a continuous function $f : D \to D$ on a dcpo D with least element (or bottom) $\perp$ has a *least fixed point* given by $\bigsqcup_{n \geq 0} f^n(\perp)$.

Given two elements x, y in a poset P, we say x is *way-below* y, or y is *way-above* x, or equivalently x *approximates* y, denoted by $x \ll y$ or $y \gg x$, if whenever $y \sqsubseteq \bigsqcup A$ for a directed set A with lub, then there is $a \in A$ with $x \sqsubseteq a$. An element $x \in P$ is *compact* if $x \ll x$. We say that a subset $B \subseteq P$ is a *basis* for P if for each $d \in P$ the set A of elements of B way-below d is directed and $d = \bigsqcup A$. We say P is *continuous* if it has a basis; it is ω-*continuous* if it has a countable basis. In any continuous poset, subsets of the form $\twoheaduparrow b = \{x | b \ll x\}$ where b belongs to a given basis give a basis of the Scott topology. An (ω-)continuous poset is (ω-)*algebraic* if it has a basis of compact elements. A dcpo is *bounded complete* if any bounded subset has a lub. By a *domain* in these notes we mean a dcpo. A dcpo is *pointed* if it has a least element. We can always add a bottom element to a domain to make it pointed.

2.2. Domains in logical form. An ω-continuous domain can be presented by a logical structure. A *continuous information system* (cf. an R-structure [110] and an abstract basis [2]) is a pair $(A, \vdash)$ where A is a non-empty countable set and $\vdash \subseteq A \times A$ is a binary relation satisfying the following, (for a finite subset $C \in \mathcal{P}_f(A)$ we write $a \vdash C$ if $\forall c \in C.\ a \vdash c$),

(i) $\forall a, b, c \in A.\ a \vdash b \vdash c \Rightarrow a \vdash c$ (transitivity), and
(ii) $\forall a \in A \forall C \in \mathcal{P}_f(A)\ [a \vdash C \Rightarrow (\exists b \in A.\ a \vdash b \vdash C)]$ (interpolation).

For any ω-continuous domain D with a countable basis B, the pair $(B, \gg)$, where $\gg$ is the restriction of the way-above relation of D to B, is a continuous information system. We think of elements of a continuous information system as propositions or assertions which tell us how to construct the associated continuous domain. In fact, from a continuous information system $(A, \vdash)$ one can construct its *rounded ideal completion* $\mathcal{I}(A)$ as follows. A *point* of $(A, \vdash)$ is a subset $x \subseteq A$ such that, (i) x is closed under entailment $(\forall a \in x \forall b \in A.\ a \vdash b \ \Rightarrow\ b \in x)$ and (ii) any finite set $C \subseteq x$ of propositions in x is derivable from some proposition in x $(\forall C \in \mathcal{P}_f(x) \exists a \in x.\ a \vdash C)$. The continuous domain $\mathcal{I}(A)$ is the set of points of $(A, \vdash)$ ordered by subset inclusion. If A has an element Δ with $a \vdash \Delta$ for all $a \in A$, then the domain will have a least element $\{\Delta\}$; hence the information system $(A, \vdash, \Delta)$ represents a pointed domain. If we think of information systems as a certain logic then the elements of the corresponding domain are in fact the *theories* of this logic.

Given information systems $(A, \vdash_A)$ and $(B, \vdash_B)$ an *approximable relation* $R : A \to B$ is a binary relation $R \subseteq A \times B$ such that, (i) $\forall a, a' \in A \forall b, b' \in B.\ a \vdash_A a' R b' \vdash_B b \Rightarrow aRb$, (ii) $\forall a \in A \forall C \in \mathcal{P}_f(B).\ (\forall c \in C.\, aRc) \Rightarrow (\exists b \in B.\ aRb \vdash_B C)$. Any continuous function $f : D \to E$ between continuous domains D and E with basis A and B, gives rise to an approximable relation $R_f : A \to B$ on the corresponding information systems defined by

$aR_f b \iff b \ll_E f(a)$. Conversely, any approximable relation between two continuous information systems induces a continuous function between the associated domains. In fact, the category of ω-continuous domains and continuous functions is equivalent with the category of continuous information systems and approximable relations.

2.3. Effectively given domains. An ω-continuous domain can be effectively presented with respect to an enumeration of a basis by requiring that the way-below relation restricted to the basis elements is recursively enumerable [123, 47]. This can be stated in terms of information systems. A continuous information system $(A, \vdash, \Delta)$ with an enumeration of its elements $A = \{a_0, a_1, a_2, \cdots\}$, where $a_0 = \Delta$, is *effectively given* with respect to this enumeration if the entailment relation $a_m \vdash a_n$ is r.e. in m and n, i.e., if the set $\{\langle m, n\rangle | a_m \vdash a_n\}$ is r.e. where $\langle ., .\rangle$ is the standard pairing function. We note that authors usually require the entailment relation to be recursive in order to obtain an effective structure on function spaces [110, 96]. An approximable relation $R : A \to B$ between effectively given information systems with enumerations $A = \{a_0, a_1, a_2, \cdots\}$ and $B = \{b_0, b_1, b_2, \cdots\}$ is *computable* if it is r.e. with respect to these enumerations, i.e., if the set $\{\langle m, n\rangle | a_m R b_n\}$ is an r.e. set. A pointed ω-continuous domain D with an enumerated basis $A = \{a_0, a_1, a_2, \cdots\}$, where $a_0 = \bot$, is *effectively given with respect to A* if its associated information system $(A, \gg, \bot)$ is effectively given. In such a domain, an element $x \in D$ is a *computable* element if the set $\{m | a_m \ll x\}$ is recursively enumerable. Equivalently, x is a computable element if it is the lub of a recursive chain of basis elements way-below it. For effectively given ω-continuous domains D and E, a continuous function $f : D \to E$ is *computable* if the approximable relation R_f on the associated information systems is computable. The lub of an effective chain of computable elements is computable and so is the least fixed point of a computable function on an effectively given domain. As seen in the last subsection, this leads to a logical presentation and an effectively given computational model.

§3. Some basic computational models.

§3. Some basic computational models. In order to construct computational models for classical spaces, we seek to embed these spaces onto the set of maximal elements of continuous domains. We will see below how this is done for the Cantor space, the real line, locally compact second countable Hausdorff spaces and complete separable[1] metric spaces, in particular separable Banach spaces. In each case we present a suitable basis so that an element of the classical space can be represented as the lub of the basis elements way-below its image under the embedding.

3.1. The domain of streams. Let Σ be a finite set. Let Σ^* and Σ^ω be the set of finite and infinite sequences over Σ respectively and let $\Sigma^\infty = \Sigma^* \cup \Sigma^\omega$ be

[1] A topological space is separable if it has a countable dense subset.

the set of streams over Σ. Then Σ^∞ with the prefix ordering is an ω-algebraic dcpo and Σ^* is the set of compact elements, which forms the canonical basis. Any infinite sequence $a_0a_1a_2\cdots$ is the lub $\bigsqcup_{n\geq 0} a_0a_1\cdots a_n$ of finite sequences. A basic Scott open set is given by $\uparrow a_0a_1\cdots a_n = \{x \in \Sigma^\infty | a_0a_1\cdots a_n \sqsubseteq x\}$. The set Σ^ω of the maximal elements with the subspace Scott topology is the Cantor space Σ^ω with the product topology when Σ is given the discrete topology.

3.2. The domain of intervals. Let $\mathbf{I}\mathbb{R}$ be the poset of bounded and closed, i.e., compact, subintervals of the real line ordered by reverse inclusion. This poset (with a top element representing the empty interval) was first proposed by Scott [105] as a data-type for real numbers. In fact, $\mathbf{I}\mathbb{R}$ is an ω-continuous bounded complete domain: The lub of any directed subset in this poset (i.e., any filtered set of compact intervals) is the intersection of the intervals. The way-below relation is given by $a \ll b$ iff b is in the interior of a. A countable basis is given by the set of all intervals with rational end points. A real number x is therefore approximated by an increasing chain, i.e., a shrinking sequence of rational nested intervals.

A basic Scott open set is given, for any open subset $O \subseteq \mathbb{R}$, by the collection $\Box O = \{a \in \mathbf{I}\mathbb{R} | a \subseteq O\}$. The maximal elements of this domain are the singleton subsets $\{x\}$ for $x \in \mathbb{R}$. The mapping $s : \mathbb{R} \to \mathbf{I}\mathbb{R}$ with $s(x) = \{x\}$ is an embedding of the real line onto the set of maximal elements as $s^{-1}(\Box O) = O$ for any open subset $O \subseteq \mathbb{R}$; this implies that the Euclidean topology coincides with the relative Scott topology on the subspace of maximal elements. Any continuous function $f : \mathbb{R} \to \mathbb{R}$ extends canonically to a Scott continuous function $\mathbf{I}f : \mathbf{I}\mathbb{R} \to \mathbf{I}\mathbb{R}$, defined on any compact interval a by $(\mathbf{I}f)(a) = f(a)$. This is the maximal extension [40] of f on $\mathbf{I}\mathbb{R}$, in other words if the continuous function $g : \mathbf{I}\mathbb{R} \to \mathbf{I}\mathbb{R}$ satisfies $g(\{x\}) = \{f(x)\}$ for all $x \in \mathbb{R}$ then $g \sqsubseteq \mathbf{I}f$. In practice, for convenience, we usually denote the maximal extension $\mathbf{I}f$ simply by f.

The continuous domain $\mathbf{I}\mathbb{R}$ can be equipped with a canonical effective structure by using the standard enumeration of rational intervals: A *computable real number* is then the lub of a shrinking sequence of rational intervals which is generated by a master program. This is also exactly how a computable number in the interval approach to computability on the real line is characterized as for example by Rogers [102, p. 371]. We can define a continuous function $f : \mathbb{R} \to \mathbb{R}$ to be *computable* if it has a computable extension $g : \mathbf{I}\mathbb{R} \to \mathbf{I}\mathbb{R}$. It is shown directly in [47] that our definition of computable real number and computable real function coincide with the well-established notion by Pour-El and Richards [101] which is equivalent to that of Weihrauch [122] and is based on the classical work of Grzegorczyk [62, 63]. In fact, it was known from the work of Stoltenberg-Hansen and Tucker [116] that the computability theory induced on the real line by its effective representation with an algebraic domain is equivalent to the classical computability theory of the real numbers and it can be shown, from results in [18], that effective presentations by algebraic and continuous domains are indeed equivalent.

We emphasize that our domain-theoretic notion of a computable real is weaker than that used in traditional constructive mathematics where a real number is given as a Cauchy sequence of rational numbers with an explicit modulus of convergence. For example, in Bishop's work [17] a real number is the limit of a sequence of rational numbers $\langle x_n \rangle_{n \geq 1}$ with $|x_n - x_m| \leq \frac{1}{n} + \frac{1}{m}$. The idea behind the domain-theoretic notion, however, is that if a real number is given as the intersection of a shrinking nested sequence of rational intervals then, examining the sequence term by term, we will eventually obtain an interval whose length is less than a given size. Of course, in practice it is very useful to know an estimate for the rate of convergence of the approximating intervals to a real number; this is in fact essential for complexity analysis. But such an estimate is almost always very conservative and to use it in order to compute a rational approximation to the real number up to a given accuracy may lead us to perform too many unnecessary steps in a computation and therefore greatly reduce the efficiency. In contrast, the domain-theoretic model provides an *incremental* framework for computation: To approximate the real number with a rational interval of length less than ε, we find the first interval in the shrinking sequence whose length is less than ε. If, subsequently, a more precise approximation is required, we will resume the search for such an interval in the sequence beginning with the earlier approximation. This is studied in detail in the context of exact real arithmetic in Section 9.

Finally we note that one can similarly construct the domain of $\mathbf{I}[x_1, x_2]$ of the compact subintervals of any real closed interval $[x_1, x_2]$. Figure 1 depicts the domain $\mathbf{I}[0, 1]$.

3.3. The upper space. For any Hausdorff space X, the *upper space* UX of X is the set of all non-empty compact subsets of X with the base of *upper topology* given by collections $\Box O = \{a \in UX \mid C \subseteq O\}$ for any open subset $O \in \Omega X$ [113]. This topology is T_0 and its specialisation ordering(see [59, p. 123] or [74, p. 45]), denoted by $\sqsubseteq$, is reverse inclusion, i.e.,

$$a \sqsubseteq b \overset{\text{def}}{\iff} \forall O \in \Omega X[a \subseteq O \Rightarrow b \subseteq O] \iff a \supseteq b.$$

$(UX, \supseteq)$ is a dcpo, in which the lub of a directed set of compact subsets is their intersection.

THEOREM 3.1. [35] *When X is locally compact, $(UX, \supseteq)$ is a continuous bounded complete dcpo with $A \ll B$ iff B is in the interior of A, and the upper*

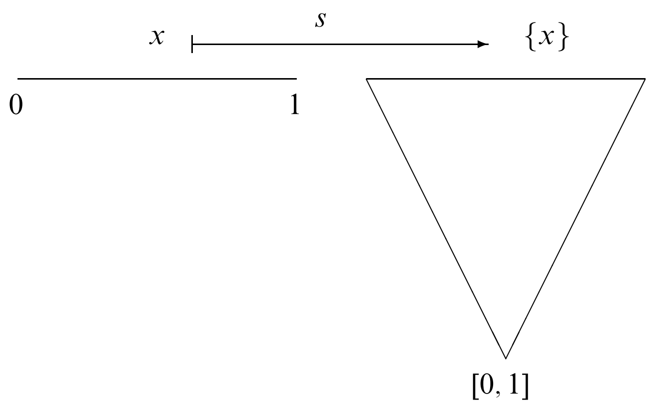

FIGURE 1. $\mathbf{I}[0,1]$.

topology coincides with the Scott topology. If X is also second countable (i.e., it has a countable basis) then $\mathbf{U}X$ is an ω-continuous dcpo.

The upper space of a second countable locally compact Hausdorff space was presented originally by the author in [35] as a simple model for computation in mathematics; we will see several applications of this model in later sections. A countable basis of $\mathbf{U}X$ is obtained as follows. We start with a countable basis of X consisting of relatively compact open subsets[2]. In fact, since a locally compact second countable space is metrizable, one can take as a basis of X the set of all finite unions of relatively compact open balls with rational radii centred at points of a countable dense subset of X. Then the collection of the finite unions of the closures of these basic open sets forms a countable basis for $\mathbf{U}X$. The maximal elements of $\mathbf{U}X$ are the singleton subsets $\{x\}$ for $x \in X$. The mapping $s : X \to \mathbf{U}X$ with $s(x) = \{x\}$ is an embedding of X onto the set of maximal elements of its upper space since $s^{-1}(\Box O) = O$. As in the case of the domain of intervals, the Hausdorff topology on X coincides with the induced relative Scott topology on the subspace of maximal elements of the upper space. Any continuous function $f : X \to X$ extends to a Scott continuous function $\mathbf{U}f : \mathbf{U}X \to \mathbf{U}X$ defined on any non-empty compact subset C by $(\mathbf{U}f)(C) = f(C)$. In fact, $\mathbf{U}$ is a functor from the category of locally compact spaces and continuous functions to the category of continuous dcpo's and continuous functions. See [35] for details. The map $\mathbf{U}f$ is the maximal extension of f and, as in the case of the domain of intervals, for convenience, we denote it simply by f. The problem of extending the domain of a map from a subspace to the space as a whole was first studied by Scott in his work on the so-called injective spaces [106]. Bounded complete continuous domains, such as $\mathbf{U}X$

[2] A set is *relatively compact* if its closure is compact.

for a locally compact space X, are precisely the so-called densely injective spaces [59, p. 127]: In the category of T_0 topological spaces and continuous maps, Y is *densely injective* if, whenever A is a dense subspace of B, any continuous map $f : A \to Y$ extends to a map $\hat{f} : B \to Y$. Escardó has recently made a new study of extension of maps for injective spaces [51].

3.4. The space of formal balls of a metric space. Topologically complete separable metrizable spaces, the so-called Polish spaces, form a fundamental class of spaces in mathematics which include separable Banach spaces and in particular separable Hilbert spaces. A Polish space is therefore the underlying topological space for a complete separable metric space. These spaces do not in general have enough compact subsets to provide a computational model as in the case of a locally compact space.

There are two constructions which embed a metric space into an algebraic dcpo [124, 19]; but these embeddings are quite involved. The question is how we can obtain a simple embedding onto the set of maximal elements of a domain. Kamimura and Tang [80] showed that the set of maximal elements of a bounded complete continuous dcpo equipped with its relative Scott topology is a Polish space (topologically complete separable metrizable space). Lawson more recently showed the following more general result.

THEOREM 3.2. [88] *If the relative Scott and Lawson topologies on the set of maximal elements of an ω-continuous dcpo coincide then the set of maximal elements equipped with this topology is a Polish space.*

He also provided an indirect and rather complicated construction via an embedding in the Hilbert cube to obtain an ω-continuous dcpo whose set of maximal elements is a given complete separable metric space; the construction is not functorial.

The following alternative construction by formal balls given in [41] is simple, direct, functorial and useful in applications. A *formal ball* [124] of a metric space (X, d) is a pair (x, r) with $x \in X$ and $r \in \mathbb{R}^+$. The space $\mathbf{B}X$ of formal balls of X is the set of all formal balls with the ordering $(x, r) \sqsubseteq (y, s) \overset{\text{def}}{\iff} d(x, y) \leq r - s$. If we denote the closed ball with centre x and radius r by $C(x, r)$, then $(x, r) \sqsubseteq (y, s)$ implies $C(x, r) \supseteq C(y, s)$. The converse is not true in general but it holds for normed vector spaces. In other words, in any normed vector space, the poset of formal balls and the set of closed balls partially ordered by reverse inclusion are isomorphic. In particular, this holds for any Banach space and consequently for Hilbert spaces.

In any metric space X, we have $(x, r) \ll (y, s) \iff d(x, y) < r - s$. Furthermore, the poset $\mathbf{B}X$ is continuous and the subsets $\twoheaduparrow(x, r) = \{(y, s) | d(x, y) < r - s\}$ form a basis of the Scott topology. A metric space X is separable iff $\mathbf{B}X$ has a countable basis (i.e., it is ω-continuous). If

X is separable, then the formal balls of the form (x, r) where x belongs to a countable dense subset of X and r is a positive rational number give a countable basis of $\mathbf{B}X$.

There is also an interesting connection between completeness of a metric space and that of a poset: a metric space X is Cauchy complete iff $\mathbf{B}X$ is directed complete. The maximal points of $\mathbf{B}X$ are precisely the elements $(x, 0)$ for $x \in X$. The mapping $i : X \to \mathbf{B}X$ with $i(x) = (x, 0)$ embeds X onto the maximal elements of the space of formal balls since $i^{-1}(\Uparrow a(x, r)) = \{y | d(x, y) < r\}$. We therefore have the following main result.

Theorem 3.3. [41] *A metric space X is complete and separable iff $\mathbf{B}X$ is an ω-continuous domain.*

We can extend $\mathbf{B}$ to a functor. Let $f : X \to Y$ be a function between two metric spaces with a Lipschitz constant $c \geq 0$, i.e., for all x, x' in X, $d(fx, fx') \leq cd(x, x')$. The collection of pairs (f, c) of a function and a Lipschitz constant forms a category with composition of morphisms given by $(f, c) \circ (g, d) = (f \circ g, cd)$. The functor $\mathbf{B}$ is defined on this category by $\mathbf{B}(f, c)(x, r) = (fx, cr)$ for every (x, r) in $\mathbf{B}X$.

This gives a simple computational model for complete metric spaces. Flagg and Kopperman [58] have developed a variant of the space of formal balls to obtain an ω-algebraic domain as a computational model for ultrametric spaces.

For a complete separable metric space X, an effective structure on $\mathbf{B}X$ induces a computability theory for metric spaces which is similar to the corresponding theory induced from representing the metric space by an ω-algebraic domain as in the work of Blanck [19]. In [46], it is shown that the vector space structure of a separable Banach space can also be extended to the domain of formal balls and be effectively presented. The computability theory induced on the separable Banach space is equivalent with the classical theory of Pour-El and Richards [101].

Interestingly, we can obtain a domain-theoretic, i.e., Tarski-like, proof of the Banach contracting mapping theorem which was first given for an embedding of a complete metric space into an algebraic domain in [19]: A contracting map on a complete metric space has a unique fixed point which is the limit of the orbit of any point in the space. In fact, if $f : X \to X$ is a contracting map with Lipschitz constant $c < 1$ on the complete metric space (X, d), then the function $g = \mathbf{B}(f, c) : \mathbf{B}X \to \mathbf{B}X$ is Scott continuous. For any $x \in X$, and $r \geq d(x, fx)/(1-c)$, we have $d(x, fx) \leq (1-c)r = r - cr$, whence $(x, r) \sqsubseteq (fx, cr) = g(x, r)$. This implies that g maps the dcpo $\uparrow(x, r)$ into itself. The least fixed point $\bigsqcup_{n \geq 0}(f^n x, c^n r) = (\lim_{n\to\infty} f^n x, 0)$ of g on $\uparrow(x, r)$ is a maximal element of $\mathbf{B}X$ and is easily seen to be the unique fixed point of g on $\mathbf{B}X$. It follows that f has a unique fixed point given by $\lim_{n\to\infty} f^n x$ for any $x \in X$; see [41] for details.

Given a metric space (X, d), one can first construct its completion $(\overline{X}, \overline{d})$ and then the continuous domain $\mathbf{B}\overline{X}$ which is isomorphic to the rounded ideal completion $\mathcal{I}(\mathbf{B}X)$ of $\mathbf{B}X$. In [66], Heckmann has shown how the metric d on X can be extended to a partial metric on $\mathbf{B}X$ and by continuity on $\mathcal{I}(\mathbf{B}X)$ such that its restriction to the maximal elements of $\mathbf{B}\overline{X} \cong \mathcal{I}(\mathbf{B}X)$ gives the completed metric $\overline{d}$ on $\overline{X}$.

§4. Dynamical systems on computational domains. In this section we investigate the relationship between dynamical systems on classical spaces and their counterparts on domain-theoretic models and show that they share many common properties. In particular we will see that a map is chaotic on the real line iff it has a chaotic extension to the domain of intervals. We will then present a domain-theoretic model for iterated function systems which gives a unifying framework for studying their various properties and provides a set of new results in the theory and applications of these systems.

A *discrete dynamical system* is given by the action of a continuous map $f : X \to X$ on a topological space X. The *orbit* of a point $x \in X$ is the sequence $\langle f^n x\rangle_{n\geq 0}$. In the theory of dynamical systems, one is interested in studying the long term behaviour of orbits. The point x is *periodic* if there exists $n \geq 1$ with $f^n x = x$. The least such n is called the *period* of x. If $n = 1$ then x is a fixed point. We say $f : X \to X$ is *chaotic* if

(i) f is topologically transitive, i.e., for any pair of non-empty open sets $a, b \subseteq X$, there exists $n > 0$ such that $f^n(a) \cap b \neq \emptyset$, and
(ii) the periodic points of f are dense in X.

If X is in fact a metric space, then it can be shown [5] that a chaotic map is *sensitive to initial conditions* i.e., there exists $\delta > 0$ such that, for any $x \in X$ and any neighbourhood N of x, there exists $y \in N$ and $n \geq 0$ such that $d(f^n(x), f^n(y)) > \delta$. These three properties are precisely the definition of a chaotic map in [25] which is widely accepted. For example, it is proved in *loc.cit.* that the map $x \mapsto 4x(1-x) : [0, 1] \to [0, 1]$ is chaotic.

In [35], it is shown that if $f : X \to X$ is chaotic on a metric space X then the maximal extension $\mathbf{U}f : \mathbf{U}X \to \mathbf{U}X$ is also chaotic; in fact any continuous extension of f will be chaotic as well. For dynamical systems on the real line, chaos is preserved and reflected on the domain of intervals:

THEOREM 4.1. *Let X be the real line or a compact interval $[a, b] \subset \mathbb{R}$. A continuous map $f : X \to X$ is chaotic iff it has a continuous chaotic extension $g : \mathbf{I}X \to \mathbf{I}X$.*

PROOF. The 'only if part' follows as in the case of the upper space. For the 'if' part, suppose $g : \mathbf{I}X \to \mathbf{I}X$ is a continuous chaotic extension of f. Then, the topological transitivity of f follows immediately from that of g. In order to show that the periodic points of f are dense, let $O \subseteq X$ be open.

We show that O contains a periodic point of f. Since $\Box O \subseteq \mathbf{I}X$ is open, there exists a periodic element $[c,d] \in \Box O$ of g. Therefore, there exists some $n > 0$ with $f^n[c,d] \subseteq g^n[c,d] = [c,d] \subset O$. Hence, by Brouwer's fixed point theorem [109], f^n has a fixed point in $[c,d]$. $\dashv$

4.1. Iterated function systems. An iterated function system (IFS) on a topological space X is given by a countable set of continuous maps $f_i : X \to X$ with $i \in I$. The IFS is denoted by $\{X; f_i | i \in I\}$. If I is finite with N elements we write it, for example, as $I = \Sigma_N = \{1, 2, \ldots, N\}$. For an IFS, one examines the behaviour of the sequence $f_{i_1} f_{i_2} \cdots f_{i_n} x$ for any initial point $x \in X$ and any code sequence $i_1 i_2 \cdots \in I^\omega$. In the past 15 years, IFS theory has been a very active area of research in fractal geometry [73, 8, 29, 83, 84, 48, 9] and has found applications in diverse areas such as mathematical finance, signal processing, computer graphics, image compression, learning automata, neural nets, statistical physics and real number computation [11, 12, 7, 10, 22, 83, 84, 15, 13, 44].

A simple example of an IFS can be constructed for the decimal representation of real numbers in $[0,1]$. Let

$$f_i : x \mapsto \frac{x+i}{10} : [0,1] \to [0,1]$$

with $i \in \{0, 1, 2, \ldots, 9\}$. Then the decimal representation of any real number in $[0,1]$ can be expressed by the IFS $\{[0,1]; f_0, \ldots, f_9\}$. In fact suppose $0.i_1 i_2 i_3 \cdots$ is an infinite sequence of digits $i_j \in \{0, 1, 2, \cdots, 9\}$ representing $x \in [0,1]$. Note that if x has a finite digit representation $0.i_1 i_2 \cdots i_n$ then it is represented by the infinite sequence $0.i_1 i_2 \cdots i_n 000 \cdots$. It is now easy to see that $\{x\} = \bigcap_{n \geq 1} f_{i_1} f_{i_2} \cdots f_{i_n}[0,1]$. In fact, for each integer $n \geq 0$, we have $0.i_1 i_2 \cdots i_n = f_{i_1} f_{i_2} \cdots f_{i_n}(0) \in f_{i_1} f_{i_2} \cdots f_{i_n}[0,1]$. Therefore, real numbers in the decimal representation can be expressed by the infinite composition of maps of the above IFS. Similarly, the binary signed representation of real numbers in $[-1,1]$ can be expressed by the IFS $\{[-1,1]; f_{-1}, f_0, f_1\}$ with $f_i : [-1,1] \to [-1,1]$ where $f_i(x) = (x+i)/2$. As we will see in the last section of this paper, iterated function systems can be used to represent other number systems and play a crucial role in one of the main approaches to exact real number computation.

4.2. Weakly hyperbolic IFS. If X is a complete metric space and the maps f_i are all contracting then the IFS is called *hyperbolic*. In the early 80's, Hutchinson [73] used the Banach fixed point theorem to deduce the existence and uniqueness of an attractor for a hyperbolic IFS, i.e., a fixed point of the contracting map

$$F : \mathbf{H}X \to \mathbf{H}X,$$

where $\mathbf{H}X$ is the set of non-empty compact subsets of X with the *Hausdorff* metric and $F(A) = f_1(A) \cup f_2(A) \cup \cdots \cup f_N(A)$. Recall that the Hausdorff metric d_H on $\mathbf{H}X$ is defined by

$$d_H(A, B) = \inf\{\delta \mid B \subseteq A_\delta \text{ and } A \subseteq B_\delta\}$$

where, for a non-empty compact subset $C \subseteq X$ and $\delta \geq 0$, the set

$$C_\delta = \{x \in X \mid \exists y \in C.\, d(x, y) \leq \delta\}$$

is the *δ-parallel body* of C. The above result easily extends to an IFS which is *eventually contracting* [57], i.e., there is some $k \geq 1$ such that the N^k maps $g_{i_1 i_2 \dots i_k} = f_{i_1} f_{i_2} \dots f_{i_k}$ are contracting for all finite sequences $i_1, i_2, \dots, i_k \in \Sigma_N^k$ of length k.

In practice, IFSs are defined on compact metric (or metrizable) spaces. Assume from now that we have an IFS $\{X; f_1, \dots, f_N\}$ on a compact metric space X. Consider the extension of the IFS $\{\mathbf{U}X; f_1, \dots, f_N\}$ on the upper space. Recall that, for convenience, we write $\mathbf{U}f$ simply as f. The map

$$F : \mathbf{U}X \to \mathbf{U}X,$$

where $F(A) = f_1(A) \cup f_2(A) \cup \cdots \cup f_N(A)$ is Scott continuous and, hence, has a least fixed point. This was first noted by Hayashi [65].

The iterates $F^m X$ generate a finitary branching tree, called the *IFS tree*, depicted in Figure 2 for $N = 2$. Each node is a subset of its parent node. For any m, $F^m X$ is in fact the union of the nodes on level m of this tree. The IFS tree, as we will see, plays a fundamental role in the domain-theoretic study of IFS.

Using the domain-theoretic model, we can generalize Hutchinson's results and deduce the existence and uniqueness of the attractor for a larger class of IFSs containing maps which are not necessarily contracting. This class is motivated by a number of applications, for example in neural nets [69, 13, 33], where one encounters IFSs which are not hyperbolic. It can arise for example in a compact interval $X \subset \mathbb{R}$ if the IFS contains a smooth map $f : X \to X$ satisfying $|f'(x)| \leq 1$ but not $|f'(x)| < 1$. We say an IFS is *weakly hyperbolic* [37] if for all infinite sequences $i_1 i_2 \dots \in \Sigma_N^\omega$ the set

$$\bigcap_{n \geq 1} f_{i_1} f_{i_2} \dots f_{i_n} X$$

contains a single point, or, equivalently, if the diameter of $f_{i_1} f_{i_2} \dots f_{i_n} X$ tends to zero as $n \to \infty$. Clearly, a hyperbolic IFS is weakly hyperbolic and, hence, we have a generalization of Hutchinson's framework on compact metric spaces. In this situation, the maps f_i are not necessarily contracting, and, hence, the map F above is not in general contracting. Furthermore, a weakly hyperbolic IFS need not be eventually contracting. Therefore, the Banach fixed point theorem can no longer be employed to prove the

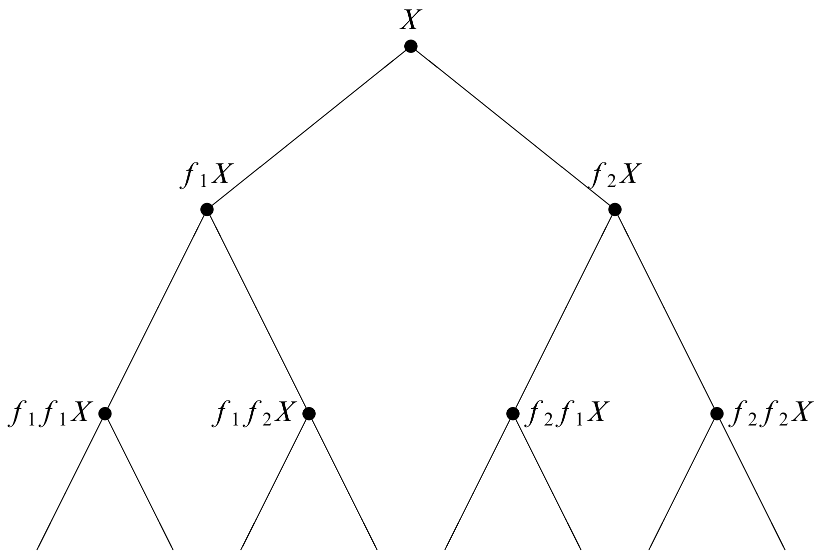

FIGURE 2. The IFS tree for $N = 2$.

uniqueness of the fixed point. This can however be proved using the *Plotkin power domain* which was originally constructed to capture the semantics of non-determinism [96].

The Plotkin power domain of an ω-continuous dcpo can be defined using the notion of finitely generable sets [111, 114]. Suppose $(D, \sqsubseteq)$ is any pointed ω-continuous dcpo and $B \subseteq D$ a countable basis for it. Consider any finitely branching tree such that each node is an element of B and each child y of any parent node x satisfies $x \sqsubseteq y$. Repetitions of elements are allowed so that an element can appear in more than one node. The set of lubs of all branches of the tree is called a *finitely generable subset* of D. We denote the set of finitely generable subsets of D by $\mathcal{F}(D)$. It is easily seen that $\mathcal{P}_f(B) \subseteq \mathcal{P}_f(D) \subseteq \mathcal{F}(D)$. For $A \in \mathcal{P}_f(B)$ and $C \in \mathcal{F}(D)$, the pre-order $\ll_{EM}$ is defined by $A \ll_{EM} C$ iff

$$\forall a \in A\, \exists\, c \in C.\, a \ll c \qquad \& \qquad \forall c \in C\, \exists a \in A.\, a \ll c.$$

This extends to a pre-order on $\mathcal{F}(D)$ by defining $C_1 \sqsubseteq_{EM} C_2$ iff for all $A \in \mathcal{P}_f(B)$ whenever $A \ll_{EM} C_1$ holds we have $A \ll_{EM} C_2$. The *Plotkin power domain* or the *convex power domain* $\mathbf{C}D$ of D is then defined to be the quotient $(\mathcal{F}(D)_{/\cong}, \sqsubseteq_{EM/\cong})$, where the equivalence relation $\cong$ on $\mathcal{F}(D)$ is given by $C_1 \cong C_2$ iff $C_1 \sqsubseteq_{EM} C_2$ and $C_2 \sqsubseteq_{EM} C_1$.

Now let D be $\mathbf{U}X$ where X is, as before, a compact metric space and $\{X; f_1, \ldots, f_N\}$ an IFS. The IFS tree generates an element of $\mathbf{CU}X$. Let

$F : \mathbf{U}X \to \mathbf{U}X$ be as before and consider the Scott continuous map $f : \mathbf{CU}X \to \mathbf{CU}X$ which is defined on the basis $\mathcal{P}_f(\mathbf{U}X)$ by the monotone map

$$\begin{array}{llll} f: & \mathcal{P}_f(\mathbf{U}X) & \to & \mathbf{CU}X \\ & \{A_j \mid 1 \leq j \leq M\} & \mapsto & \{f_i(A_j) \mid 1 \leq j \leq M, 1 \leq i \leq N\}. \end{array}$$

The set of nodes at level n of the IFS tree is then represented by $f^n\{X\}$.

For $A \in \mathbf{U}X$, let

$$\mathsf{S}(A) = \{s(x) \mid x \in A\} = \{\{x\} \mid x \in A\} \subseteq \mathbf{U}X.$$

Then, $\mathsf{S}(A)$ is a finitely generable subset of $\mathbf{U}X$ and the following result can be shown.

THEOREM 4.2. [37] *If the IFS $\{X; f_1, \ldots, f_N\}$ is weakly hyperbolic, then the two maps $F : \mathbf{U}X \to \mathbf{U}X$ and $f : \mathbf{CU}X \to \mathbf{CU}X$ have unique fixed points $A^* = \bigcap_{n\geq 0} F^n X$ and $\mathsf{S}A^*$ respectively.*

Therefore, the existence and uniqueness of the fixed point is proved domain-theoretically without finding a suitable metric and applying the contracting mapping theorem.

4.3. IFS algorithms. The IFS tree can be used to deduce an algorithm to generate the attractor of a weakly hyperbolic IFS [37] which extends the corresponding algorithm for a hyperbolic IFS [71]. We will make the assumption that, given $\varepsilon > 0$, we can determine a node for each branch of the IFS tree whose diameter is less than ε. For a hyperbolic IFS we have

$$|f_{i_1} f_{i_2} \ldots f_{i_n} X| \leq s_{i_1} s_{i_2} \ldots s_{i_n} |X|,$$

where s_i is the contractivity factor of f_i, and, therefore, we can clearly determine such a node. Another important case in which this can be done is when the IFS consists of monotone maps on $\mathbb{R}$.

Let $\varepsilon > 0$ be given and fix $x_0 \in X$. We construct a finite subtree of the IFS tree as follows. For any infinite sequence $i_1 i_2 \cdots \in \Sigma_N^\omega$, the sequence $\langle |f_{i_1} f_{i_2} \ldots f_{i_n} X| \rangle_{n\geq 0}$ is decreasing and tends to zero, and, therefore, there is a least integer $m \geq 0$ such that $|f_{i_1} f_{i_2} \ldots f_{i_m} X| \leq \varepsilon$. We truncate the infinite branch $\langle f_{i_1} f_{i_2} \ldots f_{i_n} X \rangle_{n\geq 0}$ of the IFS tree at the node $f_{i_1} f_{i_2} \ldots f_{i_m} X$ which is then a *leaf* of the truncated tree as depicted in Figure 3, and which contains the distinguished point $f_{i_1} f_{i_2} \ldots f_{i_m} x_0 \in f_{i_1} f_{i_2} \ldots f_{i_m} X$.

By König's lemma, the truncated tree has finite depth. Let L_ε denote the set of all leaves of this finite tree and let $A_\varepsilon \subseteq X$ be the set of all distinguished points of the leaves. For each leaf $l \in L_\varepsilon$, the attractor A^* satisfies $l \supseteq l \cap A^* \neq \emptyset$ and $A^* = \bigcup_{l \in L_\varepsilon} l \cap A^*$. On the other hand, for each leaf $l \in L_\varepsilon$, we have $l \cap A_\varepsilon \neq \emptyset$ and $A_\varepsilon = \bigcup_{l \in L_\varepsilon} l \cap A_\varepsilon$. It follows that $d_H(A_\varepsilon, A^*) \leq \varepsilon$. The algorithm therefore traverses the IFS tree in some specific order to obtain the set of leaves L_ε and hence the finite set A_ε which is

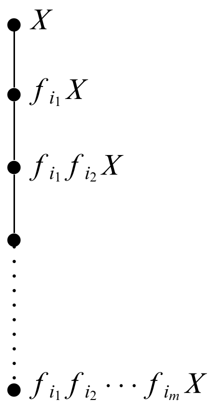

FIGURE 3. A branch of the truncated IFS tree.

the required discrete approximation. For the hyperbolic case, the complexity of the algorithm is $O(N^h)$ where h is the height of the truncated tree and is given by $h = \lceil \log(\varepsilon/|X|)/\log s \rceil$, where s is the largest contractivity factor of the maps f_i, $|X|$ is the diameter of X and $\lceil a \rceil$ is the least non-negative integer greater than or equal to a.

The domain-theoretic model has also inspired an algorithm to estimate the Hausdorff dimension of self-affine sets which has proved to be a hard problem in fractal geometry. A *self-affine* set is the attractor A of a hyperbolic IFS consisting of n affine maps $f_1, f_2, \cdots, f_n$ on $\mathbb{R}^k$. Falconer [53, 55] constructs an upper bound d for the *Hausdorff dimension* [54] of a self-affine set when the linear parts of the affine maps are non-singular and shows that for almost all choices of the translation part of these maps with respect to the Lebesgue measure on $\mathbb{R}^{nk}$ the Hausdorff dimension is actually equal to d. He also defines a number d_- which is a lower bound for the Hausdorff dimension of the self-affine set provided that the union $A = f_1(A) \cup f_2(A) \cup \cdots \cup f_n(A)$ is disjoint. However, there are no techniques to compute the values of d and d_-. In [94, 36], an algorithm is presented to generate a shrinking nested sequence of intervals, i.e., an increasing chain in the domain of intervals, with intersection $[d_-, d]$. The mth interval, which gives an approximation from below to d_- and an approximation from above to d, is obtained from the n^m compositions $f_{i_1}f_{i_2}\cdots f_{i_m}$ with $i_j \in \Sigma_n$, i.e., from the maps on the mth level of the IFS tree. In practice, this gives a reliable estimate for the dimension of a self-affine set.

§5. A computational measure theory. In the previous sections, we constructed computational models for classical spaces using continuous dcpo's and showed some applications in IFS theory. In this section, we show how to construct computational models for spaces of measures or probability distributions on classical spaces.

Recall that for a topological space X the class $\mathcal{B}X$ of *Borel subsets* of X is the smallest collection of subsets of X which contains the open subsets and is closed under taking complements and countable unions. A Borel measure is a mapping $\mu : \mathcal{B}X \to [0, \infty]$ with $\mu(\emptyset) = 0$ and $\mu(\bigcup_{i \geq 0} B_i) = \sum_{i \geq 0} \mu(B_i)$ for disjoint Borel subsets B_i $(i \geq 0)$. The set of Borel measures on X is denoted by $\mathbf{M}X$; the set of probability measures (or probability distributions), i.e., measures μ with $\mu(X) = 1$, is denoted by $\mathbf{M}^1X$. Each bounded real valued continuous function $g : X \to \mathbb{R}$ defines a functional $F_g : \mathbf{M}^1X \to \mathbb{R}$ with $F_g(\mu) = \int g \, d\mu$, the latter being the Lebesgue integral of g with respect to μ. (See Section 6 for the definition of the Lebesgue integral.) The *weak topology* on $\mathbf{M}^1X$ is the coarsest topology which makes all these functionals continuous.

A *continuous valuation* [16, 104, 86, 75, 67] is like a finite measure but is defined on open subsets. More precisely, a *continuous valuation* on a topological space Y is a mapping $v : \Omega Y \to [0, 1]$ with

(i) $v(U) + v(V) = v(U \cup V) + v(U \cap V)$.
(ii) $v(\emptyset) = 0$.
(iii) $U \subseteq V \Rightarrow v(U) \leq v(V)$.
(iv) For any directed subset $A \subseteq \Omega(Y)$ (with respect to $\subseteq$) of open sets of Y,

$$v\left(\bigcup_{O \in A} O\right) = \sup{}_{O \in A} v(O).$$

The *probabilistic power domain* $\mathbf{P}Y$ of Y is the set of continuous valuations on Y ordered pointwise, i.e., $v \sqsubseteq v' \overset{\text{def}}{\iff} v(O) \leq v'(O)$ for all open subsets $O \subseteq Y$. For any $x \in Y$ we have the point (or Dirac) valuation δ_x with $\delta_x(O) = 1$ if $x \in O$ and $\delta_x(O) = 0$ if $x \notin O$. Any linear combination $\sum_{i=1}^n r_i \delta_{x_i}$ with $x_i \in Y$ and positive numbers r_i satisfying $\sum_{i=1}^n r_n \leq 1$ gives rise to a continuous valuation; it is called a *simple valuation* as it takes only a finite number of values. In fact, any continuous valuation on a continuous domain which takes only a finite number of values is a simple valuation [81].

For any topological space Y, the poset $\mathbf{P}Y$ is a dcpo in which lubs of directed subsets are computed pointwise. If Y is an ω-continuous dcpo with a countable basis B, then $\mathbf{P}Y$ is an ω-continuous dcpo with a basis of simple valuations of the form $\sum_{i=1}^n r_i \delta_{x_i}$ with $x_i \in B$ and rational $r_i > 0$ [76]. Furthermore, Saheb-Djahromi [104], Lawson [86] and Norberg [93] have independently shown that continuous valuations on different classes of domains have unique extensions to Borel measures. It has recently been shown that any continuous valuation (and more generally any continuous

σ-finite valuation) on a continuous domain has a unique extension to a measure [3].

We now let Y be the upper space $\mathbf{U}X$ of a second countable locally compact space X. The singleton map $s : X \to \mathbf{U}X$ takes open or closed subsets of X into G_δ subsets (i.e., countable intersection of open subsets) of $\mathbf{U}X$ and Borel subsets to Borel subsets. We have:

THEOREM 5.1. [35] *For a locally compact second countable Hausdorff space X, the mapping*

$$\mu \mapsto \mu \circ s^{-1} : \mathbf{M}^1 X \to \mathbf{PU}X$$

is an embedding into the set of maximal elements of $\mathbf{PU}X$; *the image of the embedding is precisely the set of valuations $\nu \in \mathbf{PU}X$ which are supported on the set* $\max(\mathbf{U}X)$ *of the maximal elements of* $\mathbf{U}X$*, i.e.,* $\nu((\mathbf{U}X)\setminus\max(\mathbf{U}X)) = 0$.

We often identify any probability distribution with its image under the above embedding. We then get:

COROLLARY 5.2. *For any probability distribution $\mu \in \mathbf{M}^1X$, there exists an increasing chain of simple valuations $\nu_i \in \mathbf{PU}X$ with $\mu = \bigsqcup_{i\geq 0} \nu_i$.*

The above chain of simple valuations can be explicitly constructed if the measure is given on a countable basis of open sets closed under finite unions and intersections [39, 42]. In fact the result is true for *locally finite* measures, i.e., those which are finite on compact subsets; see [42]. We will illustrate this in the case of a compact metric space X. Assume $\mu \in \mathbf{M}^1X$ is a given probability measure on X. Let $\mathcal{A} = \langle A_1, A_2, \ldots, A_N\rangle$ be any ordered open covering of the compact metric space X, i.e., $A_i \subseteq X$ is open $i = 1, \ldots, N$ and $X = \bigcup_{i=1}^N A_i$. Denoting the closure of a set A by $\overline{A}$, let

$$\mu_{\mathcal{A}} = \sum_{i=1}^{N} r_i \delta_{\overline{A_i}},$$

where $r_i = \mu(A_i \setminus \bigcup_{j<i} A_j)$. Then, we have $\mu_{\mathcal{A}} \in \mathbf{P}^1\mathbf{U}X$ with $\mu_{\mathcal{A}} \sqsubseteq \mu \circ s^{-1}$. For two ordered open coverings $\mathcal{A} = \langle A_1, A_2, \ldots, A_N\rangle$ and $\mathcal{B} = \langle B_1, B_2, \ldots, B_M\rangle$, the *refinement* $\mathcal{A}\wedge\mathcal{B}$ *of* $\mathcal{A}$ *by* $\mathcal{B}$ is the ordered open covering with subsets of the form $C_{(i,j)} = A_i \cap B_j$, $1 \leq i \leq N$ and $1 \leq j \leq M$, ordered lexicographically, i.e., $(i, j) < (i', j')$ iff either $i < i'$ or $i = i'$ and $j < j'$. Then, $\mu_{\mathcal{A}} \sqsubseteq \mu_{\mathcal{A}\wedge\mathcal{B}}$. Assume $\mathcal{B}_n$ is an ordered covering of open subsets of X with diameters less than $1/n$ for $n \geq 1$. Define $\mathcal{A}_n$ for $n \geq 1$ inductively by $\mathcal{A}_1 = \mathcal{B}_1$ and $\mathcal{A}_{n+1} = \mathcal{A}_n \wedge \mathcal{B}_{n+1}$. This gives an explicit construction of an increasing chain of simple valuations and we have:

THEOREM 5.3. [39] $\mu = \bigsqcup_{m\geq 1} \mu_{\mathcal{A}_m}$.

So far, we have only considered second countable locally compact Hausdorff spaces; similar results in fact hold for the other computational models presented in this paper. More precisely, the embedding

$$\mu \mapsto \mu \circ s^{-1} : \mathbf{M}^1\mathbb{R} \to \mathbf{PI}\mathbb{R}$$

and, for any separable complete metric space X, the embedding

$$\mu \mapsto \mu \circ s^{-1} : \mathbf{M}^1 X \to \mathbf{PB}X$$

are onto the set of maximal elements of the corresponding probabilistic power domains [41]. It was conjectured in [35] that the embedding in Theorem 5.1 is onto the set of maximal elements of $\mathbf{PU}X$. This conjecture was later proved by Lawson in the following more general result.

THEOREM 5.4. [87] *If the relative Scott and Lawson topologies on the set of maximal elements of an ω-continuous domain D coincide, then the maximal elements of $\mathbf{P}D$ are precisely those continuous valuations μ on D which are supported on the set* $\max(D)$ *of maximal elements of D, i.e., $\mu(D\setminus\max(D)) = 0$; furthermore, the relative Scott and Lawson topologies on* $\max(\mathbf{P}D)$ *coincide.*

All the computational models treated in this paper satisfy the condition in Theorem 5.4. In particular, it follows that the embedding in Theorem 5.1 is also onto the set of maximal elements. We also have the following general theorem:

THEOREM 5.5. [39] *If a separable metric space is homeomorphic to a G_δ subset of an ω-continuous dcpo equipped with its Scott topology, then the space of probability measures of the metric space equipped with the weak topology is homeomorphic with a subset of the maximal elements of the probabilistic power domain of the ω-continuous dcpo.*

Therefore, in this general setting, the weak topology on the set of probability measures of a separable metric space coincides with the subspace Scott topology on a corresponding probabilistic power domain.

EXAMPLE 5.6. Let λ be the Lebesgue measure on the unit interval $[0, 1]$ and let $P : 0 = x_0 < x_1 < \cdots < x_N = 1$ be a partition of this interval with norm $\|P\| = \max_{1\leq i\leq N}(x_i - x_{i-1})$. Then

$$\mu_P = \sum_{i=1}^{N}(x_i - x_{i-1})\delta_{[x_{i-1},x_i]} \in \mathbf{PI}[0, 1],$$

with $\mu_P \sqsubseteq \lambda$. If P is refined to a partition P' then $\mu_P \sqsubseteq \mu_{P'}$. Furthermore, if $\langle P_n\rangle_{n\geq 0}$ is a refining sequence of partitions of $[0, 1]$ with $\|P_n\| \to 0$ as $n \to \infty$, then $\bigsqcup_{n\geq 0} \mu_{P_n} = \lambda$.

§**6. The generalised Riemann integral.** We use the results of the previous section to develop a theory of integration which is indeed a generalisation of the Riemann theory. We confine ourselves to integration on a compact metric space X. Assume $f : X \to \mathbb{R}$ is bounded and $\mu \in \mathbf{M}^1 X$ is a probability measure. Let $\mathbf{P}^1\mathbf{U}X$ be the subdcpo of the normalised valuations on $\mathbf{U}X$. This is again an ω-continuous dcpo with a basis of normalised simple valuations [32].

For any dcpo Y which has bottom, in particular for $\mathbf{U}X$ when X is compact, the information ordering on simple valuations in $\mathbf{P}^1 Y$ has an interesting physical interpretation. For two simple valuations

$$\mu_1 = \sum_{b \in B} r_b \delta_b \qquad \mu_2 = \sum_{c \in C} s_c \delta_c$$

in $\mathbf{P}^1 Y$, where B, C are finite subsets of Y, we have by the *splitting lemma* [76, 32]: $\mu_1 \sqsubseteq \mu_2$ iff, for all $b \in B$ and all $c \in C$, there exists a non-negative number $t_{b,c}$ such that

$$\forall b \in B \left(\sum_{c \in C} t_{b,c} = r_b \right) \qquad \forall c \in C \left(\sum_{b \in B} t_{b,c} = s_c \right)$$

and $t_{b,c} \neq 0$ implies $b \sqsubseteq c$. We can consider any $b \in B$ as a source with mass r_b, any $c \in C$ as a sink with mass s_c, and the number t_{bc} as the flow of mass from b to c. Then, the above property can be regarded as conservation of total mass.

For any simple valuation $\nu = \sum_{b \in B} r_b \delta_b \in \mathbf{P}^1\mathbf{U}X$, the *lower sum of f with respect to ν* is defined as

$$S^\ell(f, \nu) = \sum_{b \in B} r_b \inf\, f[b].$$

Similarly, the *upper sum of f with respect to ν* is defined as

$$S^u(f, \nu) = \sum_{b \in B} r_b \sup\, f[b].$$

Furthermore, for a *choice function* $\xi : B \to X$ with $\xi_b \in b$ for each $b \in B$, the sum

$$S_\xi(f, \nu) = \sum_{b \in B} r_b f(\xi_b)$$

is said to be a *generalised Riemann sum* for f with respect to ν. Note that we always have:

$$S^\ell(f, \nu) \leq S_\xi(f, \nu) \leq S^u(f, \nu).$$

If ν is replaced by a simple valuation ν' with greater information, i.e., $\nu \sqsubseteq \nu'$ then the lower sum increases and the upper sum decreases. This is exactly the situation with the Darboux sums for ordinary Riemann theory when a partition is refined. Furthermore, if $\nu_1 \ll \mu$ and $\nu_2 \ll \mu$ then there exists, by the property of the way-below relation, a simple valuation ν_3 with $\nu_1, \nu_2 \sqsubseteq \nu_3 \ll \mu$. Therefore, we have $S^\ell(f, \nu_1) \leq S^\ell(f, \nu_3) \leq S^u(f, \nu_3) \leq S^u(f, \nu_2)$. In other words, as long as we work with simple valuations way-below μ, any lower sum is less than or equal to any upper sum. This is again similar to the ordinary Riemann theory.

We then proceed to define the generalised Riemann integral as follows. The *lower R-integral* of f with respect to μ on X is

$$\mathbf{R}\underline{\int} f\,d\mu = \sup_{\nu \ll \mu} S^\ell(f, \nu).$$

Similarly, the *upper R-integral* of f with respect to μ on X is

$$\mathbf{R}\overline{\int} f\,d\mu = \inf_{\nu \ll \mu} S^u(f, \nu).$$

The lower integral is always less than or equal to the upper integral. We say f is *R-integrable* with respect to μ if these two integrals are equal in which case this common value is defined to be the R-integral of f with respect to μ.

The main results are the following [32]:

THEOREM 6.1. *R-integration has all the basic properties, including linearity, of an integral.*

THEOREM 6.2. *If $X = [0, 1] \subseteq \mathbb{R}$, then f will be R-integrable with respect to the Lebesgue measure iff it is Riemann integrable and the two integrals, when they exist, are equal.*

THEOREM 6.3. *A bounded real valued function f will be R-integrable with respect to a probability measure μ on X iff the set of discontinuities of f has μ-measure zero.*

THEOREM 6.4. *If f is R-integrable, then it is Lebesgue integrable and the two integrals are equal.*

The last three theorems generalise those of Lebesgue regarding Riemann integration early this century. R-integration has also been extended to locally finite measures (i.e., those which are finite on compact subsets) on countably based locally compact Hausdorff spaces [42].

6.1. Computation of integrals. The computational significance of the R-integral is in the following property. If $\mu = \bigsqcup_{i \geq 0} \nu_i$ and if f is continuous almost everywhere with respect to μ then,

$$S^{\ell}(f, \nu_i) \nearrow \int f\, d\mu, \qquad S^{u}(f, \nu_i) \searrow \int f\, d\mu.$$
$$S_{\xi_i}(f, \nu_i) \to \int f\, d\mu.$$

In other words, the intervals $[S^{\ell}(f, \nu_i), S^{u}(f, \nu_i)]$, $i \geq 0$, contain the Riemann sums and shrink to the value of the integral. Moreover, for any $\nu = \sum_{b \in B} r_b \delta_b \in \mathbf{P}^1\mathbf{U}X$, if the variation of f on all $b \in B$ is less than ε, i.e., $\forall b \in B \forall x, y \in b\ |fx - fy| \leq \varepsilon$, then for any Riemann sum $S_{\xi}(f, \nu)$ we have:

$$\left| \int f\, d\mu - S_{\xi}(f, \nu) \right| \leq \varepsilon. \tag{1}$$

In order to use this property in computation, we need an effective approximation of a given measure by simple valuations. This is provided by the effective version of the following proposition.

PROPOSITION 6.5. [39] *Suppose $A \subseteq \mathbf{P}^1\mathbf{U}X$ is a directed set of simple valuations. We have $\bigsqcup A \in \mathbf{M}^1X$ iff for all positive integers n and m, there exists $\sum_{c \in C} r_c \delta_c \in A$ with $\sum_{|c| \geq 1/m} r_c < 1/n$, where $|c|$ is the diameter of c.*

We now say that an increasing chain $\langle \mu_i \rangle_{i \geq 0}$ of simple valuations in $\mathbf{P}^1\mathbf{U}X$ with lub $\mu \in \mathbf{M}^1X$ is an *effective approximation* of μ if for all positive integers m and n there exists $i \geq 0$, recursively given in terms of m and n, such that $\mu_i = \sum_{c \in C} r_c \delta_c$ satisfies $\sum_{|c| \geq 1/m} r_c < 1/n$. For example, for any $\mu \in \mathbf{M}^1X$, which is given by its values on a countable basis of X closed under finite unions and intersections, the increasing chain $\langle \mathcal{A}_i \rangle_{i \geq 0}$ constructed in Theorem 5.3 is an effective approximation to μ.

Suppose $\mu \in \mathbf{M}^1X$ is effectively given with an effective approximation by a chain of simple valuations $\langle \nu_i \rangle_{i \geq 0}$. Assume that we have a Hölder continuous function $f : X \to \mathbb{R}$, i.e., there are constants $k > 0$ and $h > 0$ such that $|f(x) - f(y)| \leq k(d(x, y))^h$ for all $x, y \in X$. (If $h = 1$, the number k is a *Lipschitz constant* for f.) We can then compute the expected value of f with respect to μ up to any given accuracy as follows. Let $\varepsilon > 0$ be given. Choose the positive integers m and n with $1/m < (\varepsilon/2k)^{1/h}$ and $1/n < \varepsilon/(2k|X|^h)$, and let the integer i be such that $\nu_i = \sum_{c \in C} r_c \delta_c$ satisfies $\sum_{|c| \geq 1/m} r_c < 1/n$. We have

$$S^{\ell}(f, \mu_i) \leq \int f\, d\mu \leq S^{u}(f, \mu_i), \qquad S^{\ell}(f, \mu_i) \leq S_{\xi}(f, \mu_i) \leq S^{u}(f, \mu_i)$$

where $S_{\xi}(f, \mu_i)$ is any generalised Riemann sum for μ_i. For any $c \in C$ we have $\sup f[c] - \inf f[c] \leq k|X|^h$; whereas for $c \in C$ with $|c| < 1/m$ we have $\sup f[c] - \inf f[c] < \varepsilon/2$. Hence,

$$
\begin{aligned}
\left|\int f\,d\mu - S_\xi(f,\mu_{\mathcal{A}})\right| &\leq S^u(f,\mu_i) - S^\ell(f,\mu_i) \\
&= \sum_{c\in C}^{N} r_c(\sup f[c] - \inf f[c]) \\
&= \sum_{|c|\geq 1/m} r_c(\sup f[c] - \inf f[c]) \\
&\quad + \sum_{|c|<1/m} r_c(\sup f[c] - \inf f[c]) \\
&\leq \varepsilon/2 + \varepsilon/2 = \varepsilon.
\end{aligned}
$$

Therefore, any Riemann sum for μ_i gives the value of the integral up to ε accuracy. Hence:

THEOREM 6.6. *The expected value of any Hölder continuous function with given Hölder constants on a compact metric space can be obtained up to any given accuracy with respect to any normalised measure which has an effective approximation by an increasing chain of normalised valuations on the upper space of the metric space.*

6.2. The Lebesgue integral via the R-integral and Daniell theory. The theory of R-integration is, in a sense, dual to Lebesgue integration. In order to see this, recall the definition of the Lebesgue integral [103]. If X and Y are topological spaces then we say a map $f : X \to Y$ is *measurable* if the inverse image of any Borel subset of Y is a Borel subset of X. A *simple* function $h : X \to \mathbb{R}$ on X is one which takes only a finite set of values, i.e., one which we can write $h = \sum_{i=1}^n a_i\chi_{A_i}$ where $a_i \in \mathbb{R}$, $A_i \subseteq X$ and $\chi_A : X \to \mathbb{R}$ is the characteristic function of $A \subseteq X$ defined by $\chi(x) = 1$ if $x \in A$ and $\chi(x) = 0$ otherwise. The *Lebesgue integral* of a simple measurable function $h = \sum_{i=1}^n a_i\chi_{A_i}$ is defined by $\int h\,d\mu = \sum_{i=1}^n a_i\mu(A_i)$, with the convention $0\cdot\infty = 0$. For a positive measurable function $g : X \to \mathbb{R}$, the Lebesgue integral is defined by

$$\int g\,d\mu = \sup\left\{\int h\,d\mu \mid h \text{ simple measurable, } h \leq g\right\}.$$

The Lebesgue integral of any measurable function g is defined as $\int g\,d\mu = \int g^+\,d\mu - \int g^- d\mu$, whenever this difference exists as an extended real number, where g^+ and g^- are the positive and negative parts of g respectively.

Therefore in the Lebesgue theory the *function* which is integrated is approximated from below by *simple functions*, whereas in theory of R-integration the *measure* with respect to which integration is performed is approximated from below by *simple measures*.

Clearly the Lebesgue integral is more general than the R-integral, since the latter only exists for functions which are continuous almost everywhere with respect to the measure and these functions are, in general, a proper subclass of Lebesgue integrable functions. The question, therefore, is whether by starting with the R-integral one can obtain the Lebesgue integral of all Lebesgue integrable functions. The answer is in fact positive. We can construct the Lebesgue integral from R-integration using Daniell theory [30] as follows. The idea is that we start with the R-integral of all continuous functions on a compact metric space and then extend the R-integral stage by stage to all Lebesgue integrable functions. Let $C(X)$ denote the set of all real-valued continuous functions on the compact metric space X. Any finite measure μ on X gives rise to a continuous functional

$$\begin{array}{rccc} F_\mu : & C(X) & \to & \mathbb{R} \\ & f & \mapsto & \int f \, d\mu. \end{array}$$

One then extends F_μ to the set of all lower semi-continuous functions f on X, with $f(x) > -\infty$ for all $x \in X$, which with the pointwise ordering of functions is indeed an ω-continuous dcpo. This is achieved by continuity as any such lower semi-continuous function f is the pointwise supremum of continuous functions below it; one therefore defines

$$F_\mu(f) = \sup\{F_\mu(g) \mid f \geq g \in C(X)\}.$$

Dually, F_μ is extended to the ω-continuous dcpo of all upper semi-continuous functions with $f(x) < \infty$ for all $x \in X$. As in the general theory of Daniell integration, in the second stage, F_μ is extended to all Lebesgue integrable functions with finite integral and in the third and final stage to all Lebesgue integrable functions.

6.3. The Henstock integral. Apart from the R-integral which we have described above, there are two other notions of generalized Riemann integrals which have developed since the early sixties, namely, the McShane and the Henstock integrals [70, 60]. These are basically integrals for real valued functions on $\mathbb{R}$. Their generalisations to $\mathbb{R}^n$ also exist but they are more involved. The basic McShane integral is equivalent to the Lebesgue integral with respect to to the Lebesgue measure in the sense that a real-valued function is Lebesgue integrable with respect to to the Lebesgue measure iff it is McShane integrable. The Henstock integral (sometimes called the Henstock-Kurzweil integral) is a generalization of the McShane integral (and hence of the Lebesgue integral) in the sense that any McShane integrable function is Henstock integrable but not conversely, i.e., there are functions which are Henstock integrable but not Lebesgue integrable. The Henstock integral has the property that every continuous, almost everywhere differentiable function can be recovered by integration from its derivative.

This property, which does not hold for the Riemann or the Lebesgue integral, was historically the motivation behind the definition of this integral.

The reason the McShane and the Henstock integrals are called generalized Riemann integrals is that, similar to the ordinary Riemann integral and in contrast to the Lebesgue integral, they are defined by partitioning the domain $[a, b]$ of the integrand function f. However, a more sophisticated notion of partitioning is used as follows. A (*Perron*) *tagged subinterval* of $[a, b]$ is a pair $(x, [c, d])$, with $[c, d] \subseteq [a, b]$ such that $x \in [a, b]$ $(x \in [c, d])$. A (*Perron*) *tagged partition* of $[a, b]$ is a finite collection of (Perron) tagged subintervals $(x_i, [c_i, d_i])$ $(1 \leq i \leq n)$ of $[a, b]$ such that $[a, b] = \bigcup_i [c_i, d_i]$. A tagged partition is subordinate to a positive function $\delta : [a, b] \to \mathbb{R}^+$ if $[c_i, d_i] \subseteq (x_i - \delta(x_i), x_i + \delta(x_i))$ for all $1 \leq i \leq n$. Then one defines f to be (*Henstock*) *McShane* integrable with value k if for all $\varepsilon > 0$ there exists a positive function δ such that $|k - \sum_{i=1}^{n} (x_{i+1} - x_i) f(x_i)| < \varepsilon$ for all (Perron) tagged partitions of $[a, b]$ subordinate to δ. The resulting integration theory however is, like the Lebesgue theory, non-constructive and non-computational.

In contrast, the domain-theoretic generalization of the Riemann integral works generally for integration of functions with respect to Borel measures on Polish spaces (topologically complete separable spaces) which include locally compact second countable spaces of which $\mathbb{R}^n$ is a special case. Here, one also deals with the domain of the function rather than its range. But now one goes beyond the notion of partitions and uses finite covers by open subsets to approximate the measure by simple valuations on an ω-continuous domain as shown in the construction leading to Theorem 5.3. As we have seen in Section 6.1, this gives a notion of an effectively given measure with respect to which we can compute the integral of any Hölder continuous function up to any desired accuracy.

§7. IFS with probabilities. An IFS with probabilities is an IFS $f_i : X \to X$ $(i \in \Sigma_N = \{1, \ldots, N\})$ such that each map f_i is associated with a probability weight $p_i > 0$ with $\sum_{i=1}^{N} p_i = 1$. The *Markov operator*

$$T : \mathbf{M}^1 X \to \mathbf{M}^1 X \tag{2}$$

on the set $\mathbf{M}^1 X$ of normalised Borel measures on X takes a Borel measure $\mu \in \mathbf{M}^1 X$ to a Borel measure $T(\mu) \in \mathbf{M}^1 X$ given by

$$T(\mu)(B) = \sum_{i=1}^{N} p_i \mu(f_i^{-1}(B))$$

for any Borel subset $B \subseteq X$. Hutchinson [73] proved the existence and uniqueness of the fixed point of T for a hyperbolic IFS with probabilities on a complete metric space X. In fact, he used the Banach fixed point theorem again by showing that T is contracting with respect to the so-called *Hutchinson metric* r_H on $\mathbf{M}^1 X$ defined by

$$r_H(\mu, \nu) = \sup\left\{ \int_X f\,d\mu - \int_X f\,d\nu \mid f : X \to \mathbb{R},\right.$$
$$\left. |f(x) - f(y)| \leq d(x, y), \forall x, y \in X \right\}.$$

Using the domain-theoretic model we can deduce the existence and uniqueness of the invariant measure for weakly hyperbolic IFSs with probabilities where the Banach fixed point theorem can no longer be applied.

7.1. The invariant measure. For a weakly hyperbolic IFS with probabilities, we define the map

$$\begin{array}{rccc} H: & \mathbf{P}^1\mathbf{U}X & \to & \mathbf{P}^1\mathbf{U}X \\ & \mu & \mapsto & H(\mu) \end{array}$$

by $H(\mu)(O) = \sum_{i=1}^{N} p_i \mu(f_i^{-1}(O))$. Note that H is defined in the same way as the Markov operator T above. Then, H is Scott continuous and has, therefore, a least fixed point given by $\bigsqcup_m H^m \delta_X$, where δ_X is the bottom element of $\mathbf{P}^1\mathbf{U}X$ and the mth iteration is given explicitly by

$$H^m \delta_X = \sum_{i_1, i_2, \ldots, i_m = 1}^{N} p_{i_1} p_{i_2} \ldots p_{i_m} \delta_{f_{i_1} f_{i_2} \ldots f_{i_m} X}.$$

These iterates generate the *IFS tree with probabilities*, as in Figure 4 with $N = 2$. Each node of the tree is weighted by the product of the probabilities on the branch segments leading from the root X to that node. The mth level of the tree therefore corresponds to $H^m \delta_X$.

One can show that the least fixed point $\bigsqcup_m H^m \delta_X$ is a maximal element of $\mathbf{P}^1\mathbf{U}X$ and is, therefore, the unique fixed point; it defines a probability measure on X.

THEOREM 7.1. [37] *The map H has a unique fixed point $\bigsqcup_m H^m \delta_X$ which is the unique invariant measure of the weakly hyperbolic IFS with probabilities. The support[3] of this invariant measure is precisely the attractor of the IFS.*

We can also obtain an algorithm to compute the invariant measure of a weakly hyperbolic IFS with probabilities, extending the corresponding result for a hyperbolic IFS with probabilities in [71]. As in the deterministic case, described in Subsection 4.3 the algorithm finds all the leaves of the truncated IFS tree and, this time, computes the mass of each leaf. The set of all weighted leaves of the truncated IFS tree represents a simple valuation

[3] A point is in the support of a measure iff the measure of each open neighbourhood of the point is non-zero.

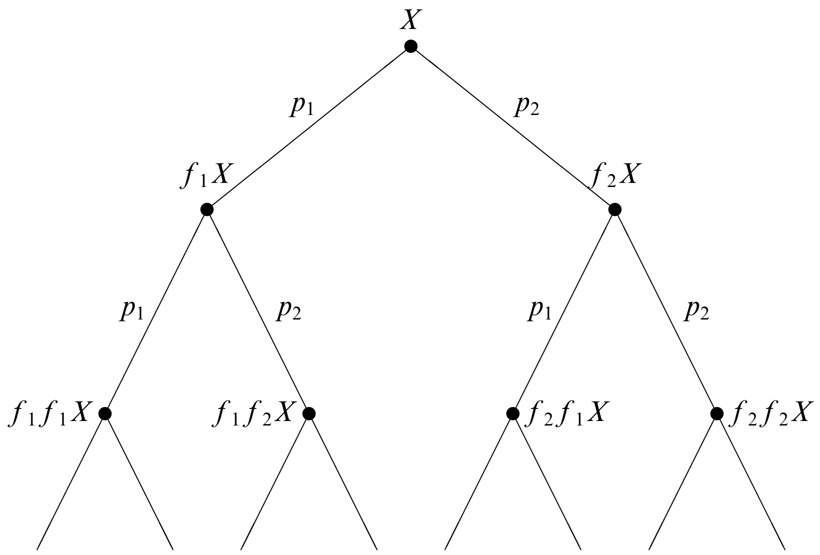

FIGURE 4. The IFS tree with probabilities for N=2.

which is a discrete approximation to the invariant measure. One can also extend the deterministic and the probabilistic algorithms to obtain their digitised versions, i.e., to plot the attractor and depict the invariant measure on a computer screen with a given resolution. See [37].

7.2. Expected values. Since the invariant measure μ is obtained as the lub of simple valuations $H^m\delta_X$, one can use the generalised Riemann integral to compute the expected value of well-behaved functions; this extends the corresponding result for a hyperbolic IFS with probabilities [71]. Suppose $g : X \to \mathbb{R}$ is continuous almost everywhere with respect to μ and let $x \in X$ be any given point. Then for the simple valuation $H^m\delta_X$ we can define the choice function ξ_m for a generalized Riemann sum by $\xi_m(f_{i_1} \ldots f_{i_m}X) = f_{i_1} \ldots f_{i_m}x$ so that

$$S_{\xi_m}(g, H^m\delta_X) = \sum_{i_1,\ldots,i_m=1}^{N} p_{i_1} \ldots p_{i_m} g(f_{i_1} \ldots f_{i_m}x).$$

It follows that

$$S_{\xi_m}(g, H^m\delta_X) \to \int g \, d\mu$$

as $m \to \infty$. If the maps f_i are contracting with contractivity factor s_i and if g satisfies a Hölder condition, then we can obtain a algorithm to calculate the integral to any given accuracy as follows [37]. Suppose there exist $h > 0$ and $k > 0$ such that g satisfies

$$|g(x) - g(y)| \leq k(d(x, y))^h$$

for all $x, y \in X$. Let $\varepsilon > 0$ be given. Then

$$\left|S(g, H^m\delta_X) - \int g d\mu\right| \leq \varepsilon \tag{3}$$

for $m = \lceil \log((\varepsilon/k)^{1/h}/|X|)/\log s \rceil$, where $s = \max_i s_i$ is the contractivity of the IFS.

In many applications, we can directly estimate the length

$$\sum_{i_1,\cdots,i_m=1}^{N} p_{i_1}\cdots p_{i_m}(\sup\ g(f_{i_1}\cdots f_{i_m}X) - \inf\ g(f_{i_1}\cdots f_{i_m}X))$$

of the interval

$$[S^\ell(g, H^m\delta), S^u(g, H^m\delta)]$$

which contains the value of the integral and, therefore, obtain a much more efficient algorithm. We will see an example of this in the next section.

One can also obtain the natural generalisations of the above IFS results for the so-called *recurrent IFS*, i.e., an IFS which is equipped with a stochastic matrix rather than just a probability vector, and also for the so-called vector recurrent IFS [34] which is the basis of Barnsley's software for fractal image compression using measures [10].

The domain-theoretic framework for IFS, as we have indicated in Section 4 and in this section, has the unifying feature that several aspects of the theory of IFS, namely (i) the proof of existence and uniqueness of the attractor of a weakly hyperbolic IFS and that of the invariant measure of a weakly hyperbolic IFS with probabilities or recurrent IFS, (ii) the algorithms to approximate the attractor and the invariant measures (iii) the complexity analysis of these algorithms, and (iv) the computation of the expected value of almost everywhere continuous functions, or Hölder continuous functions, with respect to these invariant measures, are all integrated uniformly within the model. In contrast, the classical theory uses very different, unrelated and often *ad hoc* techniques in order to obtain the corresponding results for the special class of hyperbolic IFSs.

§**8. Applications in physics.** There have been so far three areas of application of the domain-theoretic integration techniques in physics, namely in the one-dimensional random field Ising model (1dRFIM) [15, 31], in forgetful neural nets [13, 33, 43] and in periodic doubling route to chaos [56, 38, 39]. We will present the first and the third here.

8.1. The random field Ising model. The *Ising model* was introduced by Ising as a model for ferromagnetism some seventy years ago; it also describes such systems as lattice gases, binary alloys and "melting" of DNA and has been recently studied intensively in the context of complex systems [23]. The model deals with the configuration of a system of interacting objects, say two-valued spins, situated at the sites of a d-dimensional lattice (grid) in thermal equilibrium with a reservoir. The basic assumption in the Ising model is that objects interact only with their nearest neighbours as well as possibly with an external field.

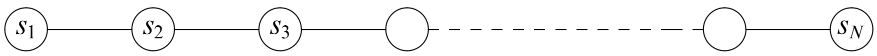

FIGURE 5. The one-dimensional $\frac{1}{2}$ spin Ising model.

Consider a one-dimensional chain of N Ising spins $\langle s_n\rangle_{n=1}^N$ with $s_n = \pm 1$ for each $1 \leq n \leq N$ as in Figure 5. Each state of the system is determined by a given set of values of the spins $s_n = \pm 1$. Assume now that there is a random magnetic field h_n at each site $n \geq 1$. For convenience, we assume the field takes only two values $h_n = \pm h$ with equal probabilities. Then, we can write the energy or the Hamiltonian of the system in any state $\langle s_n\rangle_{n=1}^N$ as

$$\mathcal{H}_N = -\sum_{n=1}^{N-1} J s_n s_{n+1} - \sum_{n=1}^{N} h_n s_n$$

where $J > 0$ is the *coupling* constant. This equation says that the only interactions which contribute to the energy of the system are between neighbouring spins on the one hand and between each spin and the magnetic field on the other hand. In statistical physics, the basic statistical information about a system is given by the canonical partition function which is defined by

$$Z = \sum_i e^{-\beta\varepsilon_i},$$

where ε_i is the energy of the state i and the summation is over all the possible states i of the system. Here, $\beta = (k_B T)^{-1}$, where T is the temperature of the system and k_B is the universal Boltzmann constant. The probability of finding the system in state i is given by $e^{-\beta\varepsilon_i}/Z$, and if $K(i)$ is the value of the physical quantity K in the state i, then the average value of K is given by

$$\langle K\rangle = \sum_i K(i) e^{-\beta\varepsilon_i}/Z.$$

For our random Ising model, the partition function is

$$Z_N = \sum_{s_1,\ldots,s_N=\pm 1} \exp\beta\left(\sum_{n=1}^{N-1} J s_n s_{n+1} + \sum_{n=1}^{N} h_n s_n\right).$$

The summation over $s_1, s_2, s_3, \ldots, s_{N-1}$ can be carried out to obtain:

$$Z_N = \sum_{s_N=\pm 1} \exp\beta\left(\xi_N s_N + \sum_{n=1}^{N-1} B(\xi_n)\right),$$

where the stochastic variable ξ_n is defined by

$$\xi_1 = h_1, \qquad \xi_n = h_n + A(\xi_{n-1}) \quad (2 \le n \le N),$$

and the real functions $A, B : \mathbb{R} \to \mathbb{R}$ are given by

$$A(x) = (2\beta)^{-1}\log(\cosh\beta(x+J)/\cosh\beta(x-J)),$$

$$B(x) = (2\beta)^{-1}\log(4\cosh\beta(x+J)\cosh\beta(x-J)).$$

Therefore, the partition function is reduced to that of a *single* spin s_N. Behn *et al* [15] have studied this stochastic equation in the past decade.

The dynamics of ξ_n, we can say, is based on the IFS with probabilities $f_+, f_- : \mathbb{R} \to \mathbb{R}$ defined by $f_+(x) = h + A(x)$ and $f_-(x) = -h + A(x)$ with probabilities $p_+ = p_- = 1/2$. Each f_σ, where $\sigma = \pm$, satisfies $|f'_\sigma(x)| \le \tanh\beta J < 1$, and is hence contracting. Also each f_σ has a unique fixed point x_σ, with

$$x_+ = -x_- = h/2 + (2\beta)^{-1}\operatorname{arcsinh}(e^{2\beta J}\sinh\beta h) > 0.$$

Furthermore $f_\sigma[x_-, x_+] \subseteq [x_-, x_+]$ for $\sigma = \pm$. The graphs of f_+ and f_- in $[x_-, x_+]$ are shown in Figure 6 on page 433.

Various physical quantities of the system can be expressed as expected values of certain continuous functions with respect to the invariant measure of the IFS. We illustrate this for the simplest case, i.e., the free energy density of the system as $N \to \infty$. In statistical physics the free energy is given by $F_N(\beta) = -\frac{1}{\beta}\log Z_N$. Therefore, the free energy density for our system is

$$\begin{aligned} f(\beta) &= \lim_{N\to\infty}\frac{F_N(\beta)}{N} = -\frac{1}{\beta}\lim_{N\to\infty}\frac{\log Z_N}{N} \\ &= -\lim_{N\to\infty}\frac{1}{N}\log(2\cosh\beta\xi_N) - \lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N-1} B(\xi_n) \\ &= -\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N-1} B(\xi_n). \end{aligned}$$

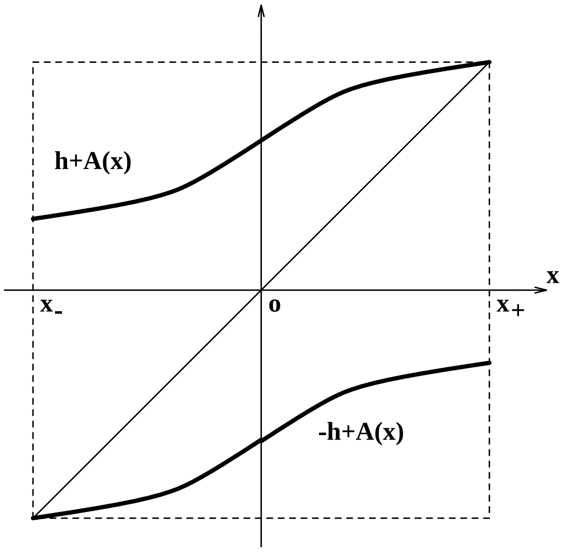

FIGURE 6. The graphs of f_+ and f_- in the interval $[x_-, x_+]$.

To evaluate the above limit we use the following theorem. Let

$$(f_1, \ldots, f_N; p_1, \ldots, p_N)$$

be a hyperbolic IFS with probabilities on a metric space X with invariant measure μ. Suppose $i_1, i_2, \ldots$ is a sequence of independent, identically distributed random variables on $\{1, 2, \ldots, N\}$ with probabilities

$$P(i_n = k) = p_k \qquad (1 \le k \le N)$$

for all $n \geq 1$. Let $x_0 \in X$, and put $x_{n+1} = f_{i_n}(x_n)$ for all $n \geq 0$. Then Elton's ergodic theorem states that the time average of any real-valued continuous function is the same as its phase average with respect to the invariant measure μ:

THEOREM 8.1. [48] *Let $g : X \to \mathbb{R}$ be a continuous function and suppose $x_0 \in X$. Then, for almost all sequences $i_1, i_2, \ldots$,*

$$\lim_{k\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} g(x_n) = \int g(x)\, d\mu(x).$$

Therefore, the free energy density is given by

$$f(\beta) = -\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} B(\xi_n) = -\int B\, d\mu.$$

A simple calculation shows that $B : [x_-, x_+] \to [x_-, x_+]$ is contracting with contractivity factor $c = B'(x_+) = \frac{1}{2}(\tanh\beta(x_+ + J) + \tanh\beta(x_+ - J)) < 1$. One can therefore use Equation (3) to compute an integer n such that $S_{\xi_n}(B, H^n\delta_X)$ gives us the value of the above integral up to any degree of accuracy ε.

However, the integer n obtained in this way is in general quite conservative for the given accuracy ε. A much better technique can be used here. Since $f_-, f_+ : [x_-, x_+] \to [x_-, x_+]$ are both monotone maps and $B : [x_-, x_+] \to [x_-, x_+]$ is piecewise monotone, one can directly compute the length

$$\sum_{i_1,\cdots,i_m=1}^{N} p_{i_1}\cdots p_{i_m}(\sup\ B([f_{i_1}\cdots f_{i_m}x_-, f_{i_1}\cdots f_{i_m}x_+]) \\ - \inf\ B([f_{i_1}\cdots f_{i_m}x_-, f_{i_1}\cdots f_{i_m}x_+])),$$

of the intervals

$$[S^{\ell}_{\xi_m}(B, H^m\delta), S^{u}_{\xi_m}(B, H^m\delta)]$$

for successive integers $m \geq 0$. When for some $m \geq 0$ the length of the interval is less than ε, we have our required estimate of the integral.

Other physical quantities can be similarly computed. For example, the magnetisation density is obtained by

$$m(\beta) = \int\int \tanh\beta(x + A(y))\, d\mu(x) d\mu(y),$$

which can be computed using the double R-integral with the two dimensional version of the Elton's ergodic theorem [31]. Numerical computation of magnetisation and entropy at finite temperatures in the model has been carried out in [14] based on the generalized Riemann integral.

8.2. Period doubling route to chaos. Feigenbaum's discovery of the period doubling route to chaos is one of the great scientific achievements of the recent decades [56]. The period doubling route to chaos is a universal way a dynamical system can become chaotic; it arises in various fields of science and engineering. The prototype of a dynamical system following this route to chaos is provided by the *Logistic family*,

$$\begin{array}{rccc} f_c : & [0,1] & \to & [0,1] \\ & x & \mapsto & cx(1-x) \end{array}$$

where c is a real number which increases from 1 to 4.

For $1 < c < 3$, the orbit $\langle f_c^n(x)\rangle_{n\geq 0}$ of any $x \in (0,1)$ converges to the unique attracting fixed point $\frac{c-1}{c}$ of f_c. At $c = c_1 = 3$, a period doubling bifurcation takes place: The attracting fixed point loses its stability and becomes repelling; at the same time an attracting periodic orbit of period two is born nearby. For $c_1 < c < c_2$, where $c_2 \approx 3.499$, the ω-limit set[4] of the orbit of any point

$$x \in (0,1) \setminus \left\{ \frac{c-1}{c} \right\}$$

is the period-two orbit. At $c = c_2$, the family goes under another period doubling bifurcation. The period-two orbit becomes repelling and at the same time an attracting period-four orbit is created nearby.

This period doubling scenario is repeated at infinitum at

$$c_1 < c_2 < c_3 < \cdots < c_n < \cdots,$$

such that at c_n $(n \geq 1)$ the attracting orbit of period 2^{n-1} becomes repelling, but in its neighbourhood an attracting orbit of period 2^n is created. We have

$$c_\infty = \lim_{n\to\infty} c_n \approx 3.569.$$

For $c > c_\infty$, the system can exhibit chaotic behaviour. This means that the ω-limit set of the orbit of a typical point is a *strange attractor*: the orbit wanders around an attracting infinite set and the orbits of two close points will eventually diverge from each other. Figure 7 depicts the attractor of the system as c increases from 1 to c_∞.

At $c = c_\infty$ the map f_{c_∞} is at the edge of chaos and is an example of a Feigenbaum map, the prototype of an infinitely renormalizable map [24, p. 113]. We will now study the dynamics of this map. For convenience, we put $f = f_{c_\infty}$. The dynamics of f is determined by the orbit $x_n = f^n x_0$

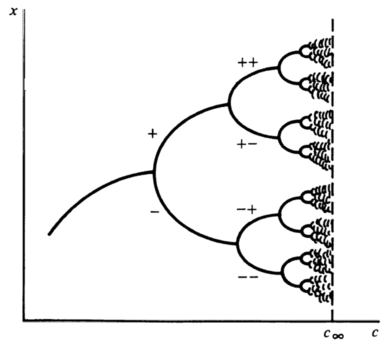

FIGURE 7. The period doubling of the attractor of the Logistic family.

$(n \geq 0)$ of the critical point $x_0 = .5$ where the derivative of f vanishes. See Figure 8.

[4]The ω-limit set of a sequence is the set of limits of all its convergent subsequences.

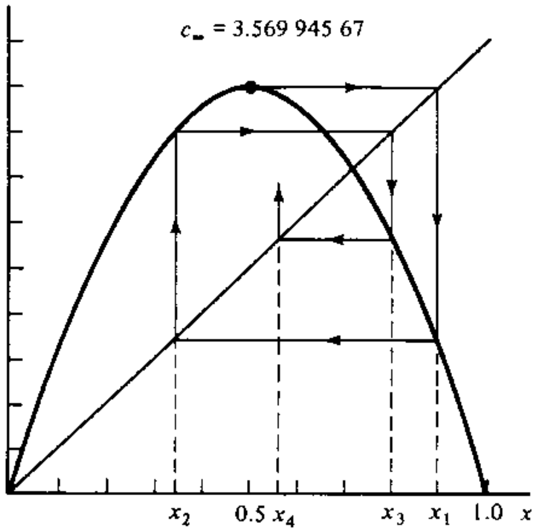

FIGURE 8. The orbit of the critical point 0.5 in the Feigenbaum map.

For each $n \geq 0$, the 2^{n+1} points $\langle x_i \rangle_{i=1}^{2^{n+1}}$ of the orbit of the critical point induce 2^n disjoint closed intervals I_j^n, with end points

$$x_j = f^j(.5) \text{ and } x_{j+2^n} = f^{j+2^n}(.5) \qquad (1 \leq j \leq 2^n)$$

such that

$$fI_j^n = I_{j+1}^n \qquad (j \geq 1, \quad \text{mod } 2^n).$$

The intervals

$$I_j^{n+1} \qquad (1 \leq j \leq 2^{n+1})$$

are nested in the intervals

$$I_j^n \qquad (1 \leq j \leq 2^n)$$

for each $n \geq 0$, as in Figure 9. This is similar to the way the Cantor set is constructed.

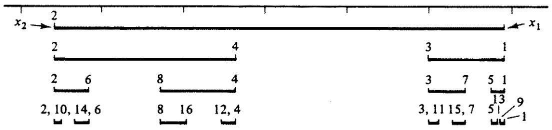

FIGURE 9. The sequence of nested intervals produced by the orbit of the critical point.

The orbit of any $x \in (0,1)$ is eventually trapped in $I^n = \bigcup_{1\le j\le 2^n} I^n_j$ for each $n \ge 0$. The length of the longest interval among I^n_j $(j = 1,\dots,2^n)$ tends to zero as $n \to \infty$. The intersection $A = \bigcap_{n\ge 0} I^n$ is a Cantor set which is the strange attractor of the system.

It is known [24, Theorem 1.6] that there exists a unique probability measure $\mu \in \mathbf{M}^1[0,1]$ which is invariant with respect to f, i.e., it is a fixed point of the map

$$\begin{array}{rccc} \mathbf{M}^1 f: & \mathbf{M}^1[0,1] & \to & \mathbf{M}^1[0,1] \\ & \mu & \mapsto & \mu \circ f^{-1}. \end{array}$$

The support of μ is the strange attractor A and μ is the unique *Bowen-Ruelle-Sinai* measure for f, i.e., it satisfies,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \phi(f^i x) = \int \phi \, d\mu,$$

for any continuous function $\phi : [0,1] \to \mathbb{R}$ and almost all $x \in [0,1]$.

Using our domain-theoretic model, we obtain this invariant measure μ as the lub of a chain of simple valuations on $\mathbf{I}[0,1]$ and present an algorithm to compute $\int \phi \, d\mu$ for a Hölder continuous function ϕ up to a given threshold of accuracy $\varepsilon > 0$. For $n \ge 1$, put

$$\nu_n = \frac{1}{2^n} \sum_{j=1}^{2^n} \delta_{I^n_j}.$$

Then, $\nu_n \sqsubseteq \nu_{n+1}$ and each ν_n is a fixed point of

$$\begin{array}{rccc} \mathbf{P}^1\mathbf{I}f: & \mathbf{P}^1\mathbf{I}[0,1] & \to & \mathbf{P}^1\mathbf{I}[0,1] \\ & \mu & \mapsto & \mu \circ f^{-1}. \end{array}$$

It follows that $\mu = \bigsqcup_{n\ge 1} \nu_n$ is also a fixed point of the above map. Since $\lim_{n\to\infty} \max_j |I^n_j| = 0$, it can be shown, by Proposition 6.5, that μ gives a probability measure on $[0,1]$. Using the uniqueness of the invariant measure, one then deduces:

THEOREM 8.2. [39] *The unique invariant measure of the Feigenbaum map f is given by $\mu = \bigsqcup_{n\ge 1} \nu_n$.*

Assume that $\phi : [0,1] \to \mathbb{R}$ is a Hölder continuous function satisfying

$$|\phi(x) - \phi(y)| \le k(|x-y|)^h$$

for all $x, y \in [0,1]$ for some $k > 0$ and $h > 0$. Let $\varepsilon > 0$. To compute $\int \phi \, d\mu$ up to ε accuracy, we obtain the least $n \ge 0$, say n_ε, such that the length of the longest interval among

$$I^n_j \quad (1 \le j \le 2^n)$$

is less than $(\varepsilon/k)^{1/h}$, i.e.,

$$|f^j(.5) - f^{j+2^n}(.5)| \leq (\varepsilon/k)^{1/h}$$

for all $j = 1, 2, \ldots, 2^n$. By the Lipschitz condition, it follows that the variation of f on the intervals $I_j^{n_\varepsilon}$ $(1 \leq j \leq 2^n)$ is less than ε. A Riemann sum for ν_{n_ε} is given by

$$S_\varepsilon = \frac{1}{2^{n_\varepsilon}} \sum_{j=1}^{2^{n_\varepsilon}} \phi(f^j(.5)).$$

It follows as in Equation (1) that $|S_\varepsilon - \int \phi \, d\mu| \leq \varepsilon$. Therefore S_ε is the required approximation. The results of this section can be generalized to a broad class of Feigenbaum maps and can be extended to some other classes of one-dimensional maps; see [39].

§9. Exact real number computation. Nearly every computer programming language provides floating-point numbers for real number computation. However, as it is well-known, they can give rise to serious problems such as round-off errors and input-error propagation, which for complex applications could indeed become critical.

There have been two main alternatives to limited precision arithmetic which have been extensively studied in the past. Interval analysis [91], by using intervals with floating endpoints, provides explicit bounds of error for all computations but it does not support exact computation. Rational arithmetic [82] performs exact computation over rational numbers by allowing unbounded integers to represent the numerator and the denominator but it cannot handle basic operations such as the square root or exponential function.

In the late 1980's two frameworks for exact real number computation were proposed. In the approach of Boehm and Cartwright [21, 20], developed and implemented recently by Valerie Menissier-Morain [90], a computable real number is approximated by B-adic numbers of the form k/B^n where B is the base, n is a natural number and k is an integer. For any basic function in analysis, a feasible algorithm has been presented in order to produce an approximation to the value of the function at a given computable real number up to any threshold of accuracy. This technique is based on the standard $\varepsilon - \delta$ analysis of elementary functions. However, the computation is not incremental in the sense that to obtain any more accurate approximation one has to compute from scratch. Furthermore, the algorithms are constructed using various different techniques and therefore, except for the simplest arithmetic operations, it is difficult to verify their correctness. Vuillemin [120], proposed a representation of computable real numbers by redundant continued fractions and, using the earlier work of Gosper [61], presented various incremental algorithms for basic arithmetic operations and some transcendental functions. This has been implemented by Lester [89]. However, this representation is rather complicated and the resulting algorithms are relatively inefficient. Nielsen and Kornerup [92] have developed a general framework for representing a real number as an infinite product of matrices or as an infinite composition of linear fractional transformations (lft).

Following Scott's idea of the domain of intervals as a representation of real numbers [105], a number of authors have worked on the notion of a real number data type. The programming language PCF (Programming Language for Computable Functions [95]) is a suitable setting to define such a data type. In his Ph.D. work [26, 27], Di Gianantonio presented an extension of PCF with a real number data type interpreted as an *algebraic* domain whose compact elements are isomorphic with the set of dyadic intervals ordered by reverse inclusion. A real number is then represented by a shrinking sequence of dyadic intervals, which can be regarded as *approximate* reals. The domain contains a representation of each real number but there are three representations for each dyadic rational. New constants are included in the language for addition and subtraction of reals by one, multiplication and division by 2, a predicate for comparison of reals with zero and a parallel conditional. The binary signed digit representation of real numbers can be embedded in this framework. He showed that any computable real function can be defined in this language, and later proved that the operational and denotational semantics are equivalent and that the language, equipped with the existential quantifier, is universal in the sense that every computable functional can be defined in it [28]. Escardó [50] developed an extension of PCF with a real number data type interpreted as the continuous domain of intervals $\mathbf{I}\mathbb{R}$. New constants are included representing contracting affine maps with rational coefficients and their left inverses as well as the predicate for comparison of intervals with zero and the parallel conditional. This language again enjoys the equivalence of its operational and denotational semantics and, equipped with the existential quantifier, is also universal [52]. Moreover, in [40], it is shown that Riemann integration can be introduced in the language. The above two approaches address the issue of formal computability rather than efficient computation. A fundamental question is whether a feasible setting for exact computation can be developed so that basic numerical calculations can be performed without round-off errors.

A practical framework for exact computation using lft's and based on domain theory has been introduced in [99, 100, 44]. It unifies the three fundamental approaches to exact computation, namely redundant digits, the B-adic numbers and the continued fractions. This approach has been effectively implemented in programming languages Caml and C++; moreover,

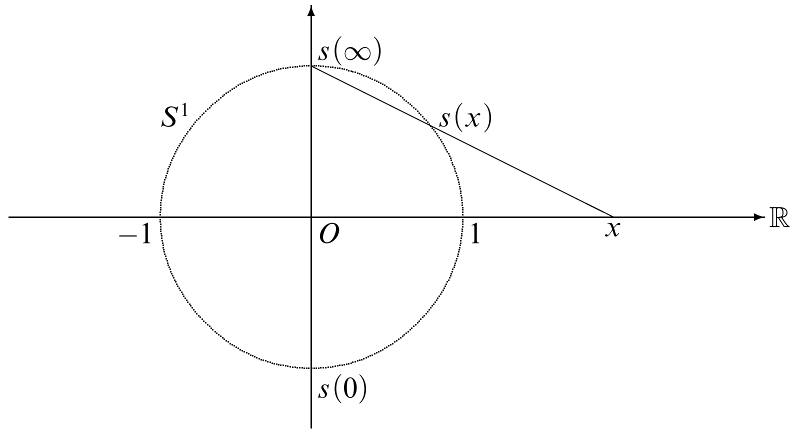

FIGURE 10. The stereographic projection.

an extension of PCF for this framework has also been developed [97]. A set of efficient and strict algorithms for elementary functions in this framework has now been developed by Potts [98]. We will explain this approach below.

9.1. The new representation of real numbers. We consider a real number as the intersection of a shrinking nested sequence of rational intervals; then the real number is computable if there is a master program which generates all these rational intervals. It is well-known that the usual predicates such as $=$, $\leq$ and $<$ on computable real numbers are not decidable. Consequently, since there is no test for zero, we have to deal with the problem of dividing say 1 by 0. Of course, test for zero is semi-decidable and one can always check before any division that the denominator is non-zero. However, this scheme is not practical as it greatly reduces the efficiency of algorithms. Therefore, any suitable framework for exact real arithmetic must allow ∞ to be the output of a program. Although the most proper framework to handle ∞ is the two-point compactification of the real line, in this article we will work with the simpler model of the extended real line $\mathbb{R}^*$ regarded as the one-point compactification of $\mathbb{R}$. A simple representation for $\mathbb{R}^*$ is the unit circle S^1 in the plane with its centre at the origin equipped with the subspace Euclidean topology of the plane. Given any point $x \in \mathbb{R}$ lying on the horizontal axis, the line joining the top point of S^1 and x intersects S^1 at a unique point $s(x)$ as in Figure 10. We define $s(\infty)$ to be the top point of S^1. Then the map $s : \mathbb{R}^* \to S^1$ is a homeomorphism and is called the stereographic projection.

The usual ordering of the real numbers induces the anti-clockwise orientation on S^1. The interval $[a, b] \subset S^1$ is defined to be the closed arc going anti-clockwise from a to b. A suitable metric on S^1 is defined as follows. For extended reals x and y which are both non-negative or both non-positive, we put

$$\rho(x, y) = \left| \frac{|x| - 1}{|x| + 1} - \frac{|y| - 1}{|y| + 1} \right|.$$

Otherwise, if x and y have different signs, then

$$\rho(x, y) = \min(\rho(x, 0) + \rho(0, y), \rho(x, \infty) + \rho(\infty, y)).$$

Similar to terms like $1/0$, we also cannot avoid expressions such as $\infty - \infty$, $0/0$ and 0^0 which must all be denoted by $\perp = \mathbb{R}^*$. This leads us naturally to the domain $\mathbf{I}\mathbb{R}^* = \{[a, b] \subset \mathbb{R}^*\} \cup \{\mathbb{R}^*\}$ of the intervals of $\mathbb{R}^*$ ordered by reverse inclusion. Any continuous function $f : \mathbb{R}^* \to \mathbb{R}^*$ has a canonical extension $\hat{f} : \mathbf{I}\mathbb{R}^* \to \mathbf{I}\mathbb{R}^*$, given by $\hat{f}(A) = f(A) = \{f(x) | x \in A\}$. For convenience, we always write $\hat{f}$ simply as f and often denote $f(A)$ simply by fA.

We will use the class of lft's or Möbius transformations with real coefficients to encode any sequence of shrinking nested intervals and, hence, any real number. The choice of lft's for this purpose is crucial to develop efficient and elegant algorithms via continued fractions for all elementary functions in this framework. In fact, mathematicians have, in the past two centuries, worked out continued fraction expansions for various functions using Padé approximants, i.e., approximation by rational functions, and studied their convergence properties [121, 4, 78].

Any continued fraction expansion of a real number can be expressed as an infinite composition of lft's of the form

$$f : x \mapsto \frac{ax + c}{bx + d} : \mathbb{R}^* \to \mathbb{R}^*, \tag{4}$$

where $\mathbb{R}^*$ is the real line extended with the point at infinity and $a, b, c, d \in \mathbb{Z}$. In fact, a continued fraction expansion

$$r = a_0 + \cfrac{b_0}{a_1 + \cfrac{b_1}{a_2 + \cfrac{b_2}{a_3 + \ddots}}}$$

of a real number r can be expressed as $r = \phi_0(r_0)$ with

$$r_0 = a_1 + \cfrac{b_1}{a_2 + \cfrac{b_2}{a_3 + \ddots}}$$

and $\phi_0(x) = a_0 + \frac{b_0}{x}$. Iterating the above scheme, we obtain $r = \phi_0\phi_1 \cdots \phi_n(r_n)$ with

$$r_n = a_{n+1} + \cfrac{b_{n+1}}{a_{n+2} + \cfrac{b_{n+2}}{a_{n+3} + \ddots}}$$

and $\phi_i(x) = a_i + \frac{b_i}{x}$ for $0 \leq i \leq n$. One can therefore identify the original continued fraction for r with the infinite composition $\phi_0\phi_1\phi_2 \cdots$. Such a representation of real numbers was already present in [120].

The set of all real lft's, denoted by $\mathbb{M}$, consists of maps f given in Equation (4) with $a, b, c, d \in \mathbb{R}$ and $ad - bc \neq 0$. An lft is a homeomorphism of $\mathbb{R}^*$; it is orientation preserving if $ad - bc > 0$ and orientation reversing if $ad - bc < 0$.

We will study the IFS $(S^1, \mathbb{M})$. First recall some elementary properties of $\mathbb{M}$ which are similar to those of complex lft's given for example in [77, Chapter 2]. Under composition of maps, $\mathbb{M}$ is a group of homeomorphisms of $\mathbb{R}^*$. If $GL(2, \mathbb{R})$ denotes the general linear group of 2×2 non-singular matrices with real coefficients, then the mapping $\Theta : GL(2, \mathbb{R}) \to \mathbb{M}$ which maps the matrix $\left(\begin{smallmatrix} a & c \\ b & d \end{smallmatrix}\right)$ to the lft ϕ with $\phi(x) = \frac{ax+c}{bx+d}$ is a group-homomorphism. The kernel K of Θ consists of all matrices of the form λI where $\lambda \neq 0$ and I is the identity matrix. Therefore, $\mathbb{M} \cong GL(2, \mathbb{R})/K$. All this means that we can identify any lft up to scaling with a 2×2 matrix. Furthermore, $\mathbb{R}^*$ can be identified with the projective real line, i.e., the set of one dimensional subspaces of $\mathbb{R}^2$. In fact, any such subspace V is spanned by a vector $v = \left(\begin{smallmatrix} k \\ l \end{smallmatrix}\right) \in V$ with $k, l \in \mathbb{R}$ not both zero. The ratio $k/l \in \mathbb{R}^*$ is independent of the choice of $v \in V$. Hence, one can identify V with k/l. The vector $\left(\begin{smallmatrix} k \\ l \end{smallmatrix}\right)$ is said to represent $x = k/l \in \mathbb{R}^*$ in *homogeneous coordinates*. The action of an lft in these coordinates is reduced to matrix multiplication. Indeed, for the lft ϕ above, we have $\phi(\frac{k}{l}) = \frac{ak+cl}{bk+dl}$, which in homogeneous coordinates can be simply written as multiplication by a representative matrix:

$$(5) \qquad \begin{pmatrix} k \\ l \end{pmatrix} \mapsto \begin{pmatrix} ak + cl \\ bk + dl \end{pmatrix} = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} k \\ l \end{pmatrix}.$$

Thus, we can freely move, on the one hand, between $k/l \in \mathbb{R}^*$ and its homogeneous representation $\left(\begin{smallmatrix} k \\ l \end{smallmatrix}\right)$ and on the other, between the lft $x \mapsto \frac{ax+c}{bx+d}$ and its matrix representation $\left(\begin{smallmatrix} a & c \\ b & d \end{smallmatrix}\right)$ which represents a linear map from $\mathbb{R}^2$ to $\mathbb{R}^2$. In both cases the representation is unique up to scaling.

A basic property of the group $\mathbb{M}$ is that for any pair of distinct triples (x_1, x_2, x_3) and (y_1, y_2, y_3) with $x_i, y_i \in \mathbb{R}^*$ $(i = 1, 2, 3)$ there exists a unique lft $\phi \in \mathbb{M}$ with $y_i = \phi(x_i)$ for $i = 1, 2, 3$. We then have:

PROPOSITION 9.1. [44] *Given two non-trivial intervals* $[p, q]$ *and* $[r, s]$ *with* $p \neq q$ *and* $r \neq s$*, there exists an lft* $\phi \in \mathbb{M}$ *with* $\phi([p, q]) = [r, s]$.

It follows that if we fix a base interval, then we can express, or encode, all other non-trivial intervals as the image of this base interval under an lft. The most efficient base interval is $[0, \infty]$ as no computation is needed to determine the lft in the proposition. If $[r, s]$ is a rational interval $[\frac{a}{b}, \frac{c}{d}]$, then the maps $x \mapsto \frac{ax+c}{bx+d}$ and $x \mapsto \frac{cx+a}{dx+b}$ have integer coefficients and map $[0, \infty]$ onto $[\frac{a}{b}, \frac{c}{d}]$ respectively reversing and preserving the orientation. Next, we would like to express shrinking sequences of nested rational intervals in terms of lft's. An lft $\phi \in \mathbb{M}$ is said to *refine* an interval $[p, q] \subset \mathbb{R}^*$ if $\phi[p, q] \subseteq [p, q]$. Consider the interval $[0, \infty]$. Let $\mathbb{M}^+ \subseteq \mathbb{M}$ be the set of lft's whose coefficients are all non-negative or, equivalently, all non-positive.

PROPOSITION 9.2. [44] $\mathbb{M}^+$ *is the set of refining lft's of* $[0, \infty]$.

Now consider $[0, \infty]$ as the base interval; we characterize the refinement of intervals when they are expressed as images of $[0, \infty]$ under lft's.

PROPOSITION 9.3. [44] *For lft's* ϕ *and* ψ *we have* $\phi[0, \infty] \supseteq \psi[0, \infty]$ *iff* $\psi = \phi\gamma$ *with* $\gamma \in \mathbb{M}^+$.

It follows that for any shrinking sequence of nested intervals $[p_0, q_0] \supseteq [p_1, q_1] \supseteq [p_2, q_2] \supseteq \cdots$ we have $[p_n, q_n] = \phi_0\phi_1 \cdots \phi_n[0, \infty]$ where $\phi_0 \in \mathbb{M}$ and $\phi_i \in \mathbb{M}^+$ for $1 \leq i \leq n$. Therefore, the sequence can be expressed as an infinite composition of lft's, or equivalently infinite product of matrices, $\phi_0\phi_1\phi_2 \cdots$. We have therefore shown that any real number can be represented as the intersection $\bigcap_{n \geq 0} \phi_0\phi_1\phi_2 \cdots \phi_n[0, \infty]$ with $\phi_0 \in \mathbb{M}$ and $\phi_i \in \mathbb{M}^+$ $(i \geq 1)$ such that ϕ_n has integer coefficients for all $n \geq 0$. If $\phi_n : x \mapsto \frac{a_n x + c_n}{b_n x + d_n}$, then in matrix notation, the real number can be expressed as the infinite product

$$\begin{pmatrix} a_0 & c_0 \\ b_0 & d_0 \end{pmatrix}\begin{pmatrix} a_1 & c_1 \\ b_1 & d_1 \end{pmatrix}\begin{pmatrix} a_2 & c_2 \\ b_2 & d_2 \end{pmatrix}\begin{pmatrix} a_3 & c_3 \\ b_3 & d_3 \end{pmatrix}\cdots.$$

We call this a *normal product*. It gives a simple representation of the computable reals: finite segments of the above matrix product give incremental interval approximations to the real number represented by the matrix product. More specifically the first matrix tells us that the result is contained in the interval $[\frac{a_0}{b_0}, \frac{c_0}{d_0}]$ or $[\frac{c_0}{d_0}, \frac{a_0}{b_0}]$ according to the sign of the determinant of the matrix. The other matrices will successively refine this interval to give better and better approximations to the real number. The first matrix is called a *sign* matrix whereas the other matrices are *digit* matrices. The *information* contained in an lft $\phi : x \mapsto \frac{ax+c}{bx+d} : \mathbb{R}^* \to \mathbb{R}^*$ is defined by $\mathsf{info}(\phi) = \phi[0, \infty]$.

9.2. Exact floating point. So far our representation allows arbitrary normal products of integer matrices $M_0M_1M_2 \cdots$ with $M_0 \in \mathbb{M}$ and $M_i \in \mathbb{M}^+$

for $i \geq 1$. This, in practice, results in some major problems. Firstly, intervals will be refined at an arbitrary rate, making any analysis of complexity of algorithms practically impossible. Secondly, matrix multiplication can quickly produce huge integers in a matrix quite disproportionate to the information contained in it.

In analogy with floating point formats, where number representations in a given base are generated by two sign symbols and a finite number of digits, we restrict the sign and digit matrices to a finite set of specific matrices. Sign matrices are rotations of S^1 whereas digit matrices are contracting maps with respect to the metric ρ on $\mathbb{R}^*$.

We start with sign matrices. The information in sign matrices must overlap and cover S^1. If we further assume that they have the same length with respect to ρ and are evenly placed on S^1, then they will be generated by rotations of S^1. The lft $\phi_{\exp i\theta} : x \mapsto \frac{x\cos\frac{\theta}{2}+\sin\frac{\theta}{2}}{-x\sin\frac{\theta}{2}+\cos\frac{\theta}{2}}$ rotates S^1 by θ. Moreover, $\phi_{\exp i\theta}$ generates a finite cyclic group iff θ is a rational multiple of 2π. Our choice will be further restricted if the lft is required to have integer coefficients.

Proposition 9.4. [44] *Suppose θ is a non-integral rational multiple of 2π. Then the lft $\phi_{\exp i\theta}$ will have integer coefficients iff $\theta = \frac{\pi}{2}$ or $\theta = \pi$.*

For $\theta = \pi$, we get the cyclic group of order 2 consisting of $\phi_{\exp i\pi} : x \mapsto -\frac{1}{x}$ and the identity lft $\mathrm{Id} : x \mapsto x$. This gives the two intervals $\mathsf{info}(\phi_{\exp i\pi}) = [\infty, 0]$ and $\mathsf{info}(\mathrm{Id}) = [0, \infty]$ which are not overlapping. For $\theta = \pi/2$ we get the cyclic group of order 4 with elements

$$\phi_{\exp\frac{i\pi}{2}} : x \mapsto \frac{x+1}{-x+1}, \quad \phi_{\exp i\pi} : x \mapsto -\frac{1}{x},$$
$$\phi_{\exp\frac{3\pi i}{2}} : x \mapsto \frac{x-1}{x+1}, \quad \mathrm{Id} : x \mapsto x,$$

with information $[1, -1], [\infty, 0], [-1, 1]$ and $[0, \infty]$ respectively. The simplest matrices representing these lft's are, respectively:

$$S_\infty = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \quad S_- = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$
$$S_0 = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \quad S_+ = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

We therefore take these as our sign matrices.

We now select an appropriate set of digit matrices from $\mathbb{M}^+$. Since compositions of digit matrices are required to represent shrinking sequences of intervals, we will look for matrices which contract distances in $[0, \infty]$ with respect to the metric ρ. Digit matrices must overlap and cover $[0, \infty]$.

Note that S_0 is a homeomorphism from $[0, \infty]$ to its image $S_0[0, \infty] = [-1, 1]$. Let $\phi \in \mathbb{M}^+$ and consider the restriction $\phi : [0, \infty] \to [0, \infty]$. Then

$S_0\phi S_0^{-1}$ is a homeomorphism from $[-1, 1]$ onto itself. For $x, y \in [0, \infty]$ we have $\rho(x, y) = |S_0(x) - S_0(y)|$ and we get:

PROPOSITION 9.5. [44] *The map* $\phi : [0, \infty] \to [0, \infty]$ *is contracting with respect to the* ρ*-metric iff* $S_0\phi S_0^{-1} : [-1, 1] \to [-1, 1]$ *is contracting with respect to the Euclidean metric.*

It follows that for any base $b > 1$, the signed digit representation on $[-1, 1]$ in base b induces via the homeomorphism S_0 a suitable set of digit matrices in $\mathbb{M}^+$.

The signed digit system in base $b > 1$ in $[-1, 1]$ is generated by an IFS on $[-1, 1]$ with contracting maps

$$\begin{array}{rccc} f_k : & [-1, 1] & \mapsto & [-1, 1] \\ & x & \mapsto & \dfrac{x + k}{b} \end{array}$$

with $k \in \mathrm{Dig}(b) = \{-b + n, b - n | n \in \mathbb{N}, 1 \leq n \leq \lfloor b \rfloor\}$, where $\lfloor b \rfloor$ is the integral part of b. Here, b can be allowed to be a rational or an irrational number. The case $b = 3/2$ was considered by Brouwer and the case $b = \frac{1+\sqrt{5}}{2}$, the golden ratio, has been studied by Di Gianantonio [26]. We now define the digit matrices in base b as the IFS on $[0, \infty]$ with ρ-contracting maps:

$$D_k = S_0^{-1} f_k S_0 = \begin{pmatrix} 1 + b + k & -1 + b + k \\ -1 + b - k & 1 + b - k \end{pmatrix}.$$

For example, for base 2, we have the four sign matrices S_+, S_∞, S_{-1} and S_0 together with the three digit matrices which had already appeared in the work of Nilsen and Kornerup [92]

$$D_{-1} = \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix} \quad D_0 = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix} \quad D_1 = \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}.$$

Exact floating point in base b is defined as the representation of real numbers by infinite composition of lft's, or, equivalently, infinite product of matrices, such that the first matrix is one of the sign matrices above and the subsequent matrices are digit matrices. For each finite composition $D_{k_1} D_{k_2} \cdots D_{k_n}$ of digit matrices we have:

$$S_0 D_{k_1} D_{k_2} \cdots D_{k_n}[0, \infty] = f_{k_1} f_{k_2} \cdots f_{k_n}[-1, 1].$$

Therefore, for every infinite composition of digit matrices, we obtain

$$\bigcap_{n \geq 0} S_0 D_{k_1} D_{k_2} \cdots D_{k_n}[0, \infty] = \bigcap_{n \geq 0} f_{k_1} f_{k_2} \cdots f_{k_n}[-1, 1].$$

This gives us:

PROPOSITION 9.6. [44] *A real number with signed digit expansion* $.k_1k_2k_3\cdots$ $($*with* $k_j \in \mathrm{Dig}(b)$ *for* $j \geq 1)$ *is represented in exact floating point by the infinite product*

$$S_0 D_{k_1} D_{k_2} D_{k_3} \cdots .$$

9.3. Computation of elementary functions. Algorithms for computing elementary functions in this framework were first developed in [99, 100] using lft's with two arguments as proposed initially by Gosper [61]. Consider a map $f : S^1 \times S^1 \to S^1$ where

$$f(x, y) = \frac{axy + cx + ey + g}{bxy + dx + fy + h}.$$

In the same way that in homogeneous coordinates an lft is represented by a linear map from $\mathbb{R}^2$ to $\mathbb{R}^2$ as in Equation (5), an lft with two arguments is represented in homogenous coordinates by a bi-linear map from $\mathbb{R}^2 \times \mathbb{R}^2$ to $\mathbb{R}^2$, which we call a *tensor* [117]:

$$\begin{aligned}\left[\begin{pmatrix} x \\ x' \end{pmatrix}, \begin{pmatrix} y \\ y' \end{pmatrix}\right] &\mapsto \begin{pmatrix} axy + cxy' + ex'y + gx'y' \\ bxy + dxy' + fx'y + hx'y' \end{pmatrix} \\ &= \begin{pmatrix} a & c & e & g \\ b & d & f & h \end{pmatrix} \left[\begin{pmatrix} x \\ x' \end{pmatrix}, \begin{pmatrix} y \\ y' \end{pmatrix}\right].\end{aligned}$$

The information in f is the interval $\mathsf{info}(f) = f([0,\infty],[0,\infty])$. By choosing the coefficients of the above appropriately, we can obtain all basic arithmetic operations in terms of a tensor; for example, addition corresponds to choosing $c = e = h = 1$ and $a = b = d = f = g = 0$. The value of $f(x, y)$, for $x = D_{i_1} D_{i_2} \cdots$ and $y = D_{j_1} D_{j_2} \cdots$, is represented by $T(D_{i_1} D_{i_2} \cdots, D_{j_1} D_{j_2} \cdots)$. This is a simple example of an *expression tree* where T is the root and there are two possibly infinite branches corresponding to x and y. To evaluate the expression tree one *absorbs* information from the two input arguments, i.e., the two branches of the tree, into the tensor and *emits* a sign matrix followed by digit matrices from the tensor as output. The absorption and emission rules reflect the composition of lft's with two, one or zero arguments corresponding, respectively, to tensors, matrices and vectors. Since at each step one can absorb information either from the left or from the right argument of a tensor, a fair strategy is employed which gives a sequential algorithm for evaluating the expression tree; see [100]. The evaluation of the expression tree is performed in a *lazy way*, i.e., new information from the input is extracted only if it is needed to evaluate the expression tree up to a given accuracy.

One can construct continued fraction expansions with integer coefficients for all algebraic and transcendental functions [121, 4, 78]. For example, the function arctan has the following expansion

$$\arctan x = \cfrac{x}{1 + \cfrac{\frac{x^2}{3}}{1 + \cfrac{\frac{4x^2}{15}}{1 + \ddots}}},$$

which can be transformed into

$$\arctan x = \prod_{n=1}^{\infty} \begin{pmatrix} 0 & x \\ n^2 x & 2n-1 \end{pmatrix}.$$

This is an infinite composition of lft's with non-negative coefficients but now each lft has x as a parameter, i.e., it is a function of two arguments. In fact in homogeneous coordinates we have:

$$\begin{pmatrix} 0 & x \\ n^2 x & 2n-1 \end{pmatrix} \begin{pmatrix} y \\ 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ n^2 & 0 & 0 & 2n-1 \end{pmatrix} \left[\begin{pmatrix} x \\ 1 \end{pmatrix}, \begin{pmatrix} y \\ 1 \end{pmatrix} \right].$$

To compute such functions, we will therefore need lft's of two arguments. The above expansion of $\arctan x$, for $x \in [0, \infty]$, is reduced to the following expression tree:

$$\begin{aligned} &\arctan x \\ &= \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} \left[x, \begin{pmatrix} 0 & 1 & 0 & 0 \\ 4 & 0 & 0 & 3 \end{pmatrix} \left[x, \begin{pmatrix} 0 & 1 & 0 & 0 \\ 9 & 0 & 0 & 5 \end{pmatrix} [x, \cdots] \right] \right]. \end{aligned}$$

See [98] for a set of algorithms for elementary functions in this framework. These algorithms are efficient in time but their space complexity in general grows exponentially. More specifically, Heckmann [68] has shown that the size of integers in a tensor after a total number of n absorptions or emissions grows as $O(2^n)$.

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DEPARTMENT OF COMPUTING
IMPERIAL COLLEGE OF SCIENCE, TECHNOLOGY AND MEDICINE
180 QUEEN'S GATE
LONDON SW7 2BZ, UK

E-mail: ae@doc.ic.ac.uk