### Fundamental Theorems of Calculus

<table>
<thead>
<tr>
<th>Interpretation</th>
<th>Integral of derivative across solid</th>
<th>Integral across boundary</th>
<th>Line integrals, $C$ a path from $P$ to $Q$</th>
<th>Green’s Theorem (Line integral)</th>
<th>Stokes Theorem (Line integral)</th>
<th>Green’s Theorem (Flux version)</th>
<th>Divergence Theorem</th>
</tr>
</thead>
<tbody>
<tr>
<td>The change in the value of $f$ between $a$ and $b$ is the sum of the instantaneous changes over the interval $[a, b]$.</td>
<td>$\int_a^b f'(x) , dx$</td>
<td>$\int_C \nabla f \cdot dr$</td>
<td>$\int_C (F \cdot dr)$</td>
<td>$\int_D \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} , dA$</td>
<td>$\int_{\partial D} F \cdot dr$</td>
<td>$\int_S \text{curl}(F) \cdot dS$</td>
<td>$\int_{\partial W} F \cdot dS$</td>
</tr>
<tr>
<td>The change in $f$ from $P$ to $Q$ is the sum of the instantaneous changes over <em>any</em> path from $P$ to $Q$.</td>
<td>$\int_C f , dr$</td>
<td>$\int_C F \cdot dr$</td>
<td>$\int_{\partial D} F \cdot dr$</td>
<td></td>
<td>$\int_S \text{curl}(F) \cdot dS$</td>
<td>$\int_{\partial W} \text{curl}(F) \cdot dS$</td>
<td></td>
</tr>
<tr>
<td>The work done by $F$ along $\partial D$ (the line integral) is the sum of the instantaneous spins over $D$, a region in the plane.</td>
<td>$\int_D \partial F_2 , dx - \partial F_1 , dy , dA$</td>
<td>$\int_D \text{curl}(F) \cdot dA$</td>
<td>$\int_S \text{curl}(F) \cdot dS$</td>
<td></td>
<td></td>
<td>$\int_{\partial W} \text{curl}(F) \cdot dS$</td>
<td></td>
</tr>
</tbody>
</table>

1 We can obtain a nice interpretation of the “sum of instantaneous spins over $S$” by noting that the right side of Stokes’ is the flux of curl($F$) through $S$ and recalling that curl($F$) is perpendicular to the plane of rotation of $F$. 

---
Some Applications of the Divergence Theorem

Recall that we can use Green’s Theorem in a clever way to compute the area of a region \( D \subset \mathbb{R}^2 \). Similarly, we can use the Divergence Theorem to compute the volume of a region \( W \subset \mathbb{R}^3 \), provided the boundary \( \partial W \) is smooth. To see this, let \( F(x,y,z) = (x,y,z) \). Then

\[
\int_{\partial W} F \cdot dS = \iiint_W \text{div}(F) dV = \iiint_W \left( \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z) \right) dV = \iiint_W 3dV = 3 \iiint_W 1dV.
\]

Dividing both sides by 3, we see that

\[
\text{Vol}(W) = \frac{1}{3} \int_{\partial W} F \cdot dS.
\]

**Example.** (Section 18.3, Exercise 21) Use the above equation to compute the volume of the unit ball \( B = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \leq 1 \} \).

*(Solution)* We have \( F = (x,y,z) \) and

\[
\partial B = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \leq 1 \}.
\]

We can parametrize \( \partial B \) by

\[
G(\theta, \phi) = (\cos \theta \sin \phi, \sin \theta \sin \phi, \cos \phi),
\]

where \( 0 \leq \theta \leq 2\pi \) and \( 0 \leq \phi \leq \pi \). Then

\[
\frac{\partial G}{\partial \theta} = (-\sin \theta \sin \phi, \cos \theta \sin \phi, 0) \quad \text{and} \quad \frac{\partial G}{\partial \phi} = (\cos \theta \cos \phi, \sin \theta \cos \phi, -\sin \phi),
\]

so

\[
\mathbf{N}(\theta, \phi) = \begin{vmatrix}
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
-\sin \theta \sin \phi & \cos \theta \sin \phi & 0 \\
\cos \theta \cos \phi & \sin \theta \cos \phi & -\sin \phi
\end{vmatrix} = \left( -\cos \theta \sin^2 \phi, -\sin \theta \sin^2 \phi, -\sin^2 \theta \sin \phi \cos \phi - \cos^2 \theta \sin \phi \cos \phi \right) = \left( -\cos \theta \sin^2 \phi, -\sin \theta \sin^2 \phi, -\sin \phi \cos \phi \right).
\]

Notice that this normal vector points inward; to obtain an outward-pointing normal vector we let

\[
\mathbf{N}(\theta, \phi) = \left( \cos \theta \sin^2 \phi, \sin \theta \sin^2 \phi, \sin \phi \cos \phi \right).
\]

Then we compute

\[
\text{Vol}(B) = \frac{1}{3} \int_{\partial B} F \cdot dS = \frac{1}{3} \int_0^\pi \int_0^{2\pi} \langle \cos \theta \sin \phi, \sin \theta \sin \phi, \cos \phi \rangle \cdot \mathbf{N}(\theta, \phi) d\theta d\phi
\]
\[
\begin{align*}
&= \frac{1}{3} \int_{0}^{\pi} \int_{0}^{2\pi} \left[ (\cos \theta \sin \phi)(\cos \theta \sin^2 \phi) + (\sin \theta \sin \phi)(\sin \theta \sin^2 \phi) \\
&\quad + (\cos \phi)(\sin \phi \cos \phi) \right] d\theta d\phi \\
&= \frac{1}{3} \int_{0}^{\pi} \int_{0}^{2\pi} \left[ \cos^2 \theta \sin^3 \phi + \cos^2 \theta \sin^3 \phi + \sin \phi \cos^2 \phi \right] d\theta d\phi \\
&= \frac{1}{3} \int_{0}^{\pi} \int_{0}^{2\pi} \left[ \sin^3 \phi + \sin \phi \cos^2 \phi \right] d\theta d\phi \\
&= \frac{1}{3} \int_{0}^{\pi} \int_{0}^{2\pi} \sin \phi (\sin^2 \phi + \cos^2 \phi) d\theta d\phi \\
&= \frac{1}{3} \int_{0}^{\pi} \int_{0}^{2\pi} \sin \phi d\theta d\phi = \frac{2\pi}{3} \int_{0}^{\pi} \sin \phi d\phi = \frac{2\pi}{3} [\cos \phi]_{0}^{\pi} = \frac{2\pi}{3} (1 - (-1)) = \frac{4\pi}{3}.
\end{align*}
\]

Thankfully, this agrees with the known volume of \( B \). \( \diamondsuit \)

**Example.** (Chapter 18 Review, Exercise 33) The velocity vector field of a fluid (in meters per second) is

\[
\mathbf{F}(x, y, z) = \langle x^2 + y^2, 0, z^2 \rangle.
\]

Let \( W \) be the region between the hemisphere

\[
S = \{ (x, y, z) : x^2 + y^2 + z^2 = 1, \ z \geq 0 \}
\]

and the disk \( D = \{ (x, y, 0) : x^2 + y^2 \leq 1 \} \) in the \( xy \)-plane. (Notice that our \( S \) is different from what’s written in the textbook; the book has a typo.)

(a) Show that the flow rate across \( D \) is zero.

(b) Use the Divergence Theorem to compute the flow rate across \( S \), oriented with outward-pointing normal.

*(Solution)*

(a) Note that we weren’t given an orientation for the normal vector to \( D \), but that orientation shouldn’t matter since the flow rate is supposed to be zero. We’ll see that it really doesn’t matter. We can parametrize \( D \) by

\[
G(r, \theta) = (r \cos \theta, r \sin \theta, 0),
\]

where \( 0 \leq r \leq 1 \) and \( 0 \leq \theta \leq 2\pi \). Then

\[
\frac{\partial G}{\partial r} \times \frac{\partial G}{\partial \theta} = \begin{vmatrix}
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
\cos \theta & \sin \theta & 0 \\
-r \sin \theta & r \cos \theta & 0
\end{vmatrix} = \langle 0, 0, r \rangle,
\]

so \( \mathbf{N}(r, \theta) = \langle 0, 0, \pm r \rangle \), with the sign of the last component depending on the orientation of \( D \). But

\[
\mathbf{F}(G(r, \theta)) = \langle r^2 \cos^2 \theta + r^2 \sin^2 \theta, 0, 0 \rangle = \langle r^2, 0, 0 \rangle,
\]

so

\[
\text{flow rate} = \iint_{D} \mathbf{F} \cdot d\mathbf{S} = \int_{0}^{2\pi} \int_{0}^{1} \langle r^2, 0, 0 \rangle \cdot \langle 0, 0, \pm r \rangle dr d\theta = 0.
\]
We see that the sign of $r$ in $N(r, \theta)$ does not matter. Altogether, our computation says that the net flow through the unit disk $D$ is zero; this does not mean that there is no flow through $S$. It’s useful to note that most of the work we did here was unnecessary. At any point $(x, y, 0) \in D$, $F(x, y, 0) = (x^2 + y^2, 0, 0)$. This vector is tangent to $D$ and is thus contributing nothing to the flux through $D$. Concretely, $F \cdot n = 0$, so
\[
\int_D F \cdot dS = \int_D F \cdot ndS = 0.
\]

(b) If we orient $\partial W$ with an outward-pointing normal vector, then
\[
\partial W = S \cup D,
\]
where $S$ has an outward-pointing normal and $D$ has a downward-pointing normal. Then for any vector field $F$ (not just the $F$ we care about here),
\[
\int_{\partial W} F \cdot dS = \int_S F \cdot dS + \int_D F \cdot dS.
\]
This equation holds for our particular $F$ and in words says “the flow rate of $F$ out of $W$ equals the flow rate of $F$ across $S$ plus the flow rate of $F$ across $D$.” Since we’ve already computed the latter value to be zero, this means
\[
\int_S F \cdot dS = \int_{\partial W} F \cdot dS.
\]
According to the Divergence Theorem this becomes
\[
\int_S F \cdot dS = \int_{\partial W} F \cdot dS = \iiint_W \text{div}(F)dV.
\]
Then we have
\[
\int_S F \cdot dS = \iiint_W (2x + 0 + 2z) dV
= \int_0^{\pi/2} \int_0^{2\pi} \int_0^1 (2\rho \cos \theta \sin \phi + 2\rho \cos \phi)\rho^2 \sin \phi \, d\rho d\theta d\phi
= 2 \int_0^{\pi/2} \int_0^{2\pi} \int_0^1 \rho^3 (\cos \theta \sin^2 \phi + \cos \phi \sin \phi) \, d\rho d\theta d\phi
= \frac{2}{4} \int_0^{\pi/2} \int_0^{2\pi} (\cos \theta \sin^3 \phi + \cos \phi \sin \phi) \, d\theta d\phi
= \frac{2\pi}{2} \int_0^{\pi/2} \cos \phi \sin \phi d\phi = \pi \int_0^1 u du = \frac{\pi}{2}.
\]
So the flow rate out of $W$ is $\frac{\pi^2}{4}$, and all of this flow is occurring through $S$. 

\[\diamondsuit\]