This week we’ll finish discussing the double integral for non-rectangular regions (see the last few pages of the week 1 notes) and then we’ll touch briefly on triple integration, which doesn’t require a great mental leap after double integration. After this we’ll quickly recall the polar coordinate system for the plane before describing how to compute double integrals in this coordinate system.

A Triple Integral

Just as with double integrals, we define triple integration first over rectangular regions (that is, regions which are rectangular prisms) before moving on to more general regions. Also as with double integrals (and quadruple integrals, quintuple integrals, etc.), we compute triple integrals as **iterated integrals**. Here we’ll work one example of a triple integral that demonstrates a nice application: finding mass. Suppose we have a solid that takes up the region $D$, and that this solid has density function $\delta(x, y, z)$. That is, at each point $(x, y, z) \in D$ the solid has density $\delta(x, y, z)$. Then the mass of the solid is given by

$$M = \iiint_D \delta(x, y, z)\,dV.$$

We’ll use this to find the mass of a particular solid.

**Example.** Find the mass of the solid that has density $\delta(x, y, z) = xz$ and is enclosed by $y = 9 - x^2$ (for $x \geq 0$), $x = 0$, $y = 0$, $z = 0$, and $z = 1$.

**(Solution)** Since we require that $y \geq 0$, we have $x \leq 3$. So the region in question is described by

$$\mathcal{R} = \{(x, y, z) : 0 \leq x \leq 3, 0 \leq y \leq 9 - x^2, 0 \leq z \leq 1\}.$$

Here’s a plot of the region:
Having described our region, we can now compute its mass with an integral:

\[ M = \iiint_D \delta(x, y, z) dV = \int_0^1 \int_0^3 \int_0^{9-x^2} xyz \, dz \, dx \, dy = \int_0^1 \int_0^3 \left( \int_0^{9-x^2} xyz \, dz \right) \, dx \, dy = \int_0^1 \int_0^3 \left( \int_0^{9-x^2} z \right) \, dx \, dy = \int_0^1 \int_0^3 \left( \int_0^{9-x^2} z \right) \, dz = \frac{81}{4} \left( \frac{z^2}{2} \right)_0^1 = \frac{81}{4} \left( \frac{9}{2} - \frac{1}{2} \right) = \frac{81}{8}. \]

**Polar Coordinates**

Next, let’s put a new coordinate system on \( \mathbb{R}^2 \). Suppose I give you a radius \( r > 0 \) and an angle \( 0 \leq \theta < 2\pi \). Then you can draw a line segment from the origin of length \( r \) which makes an angle of \( \theta \) with the \( x \)-axis and arrive at a point in the plane, as below:

The rectangular (that is, \( x - y \)) coordinates of this point are then \( x = r \cos \theta \) and \( y = r \sin \theta \).

On the other hand, if we start with a point \( (x, y) \in \mathbb{R}^2 \) we can draw a line segment from the origin to \( (x, y) \). This segment will have length \( r = \sqrt{x^2 + y^2} \) and will make some angle \( \theta \) with the \( x \)-axis. From the figure above we see that we have a right triangle; the side length opposite \( \theta \) is \( y \) and the side length adjacent to \( \theta \) is \( x \), so \( \tan \theta = y/x \). That is, \( \theta = \tan^{-1}(y/x) \), where we’re assuming \( x, y > 0 \). All together, our coordinate changes are given by:

<table>
<thead>
<tr>
<th>Polar-to-rectangular</th>
<th>Rectangular-to-polar</th>
</tr>
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<tbody>
<tr>
<td>( x = r \cos \theta )</td>
<td>( r^2 = x^2 + y^2 )</td>
</tr>
<tr>
<td>( y = r \sin \theta )</td>
<td>( \theta = \tan^{-1}(y/x) )</td>
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**Integration in Polar Coordinates**

Now that we have a new coordinate system for \( \mathbb{R}^2 \), we’d like to describe double integration in this coordinate system. When we defined double integration in rectangular coordinates, we
started by considering rectangular regions. In polar coordinates, our most basic regions are **polar rectangles**, which are wedges of annuli, such as what you see below:

These polar rectangles admit set descriptions similar to those for standard rectangles. For example, the above wedge is given by

\[ \mathcal{R} = \left\{ (r, \theta) : \frac{1}{2} \leq r \leq 1, \frac{\pi}{4} - 0.2 \leq \theta \leq \frac{\pi}{4} + 0.2 \right\}. \]

If we have a function \( f(r, \theta) \) defined on this region, we can then define \( \iint_{\mathcal{R}} f(r, \theta) dA \) in the same way we defined the double integral for rectangular coordinates: by subdividing our region into smaller polar rectangles and approximating the value of \( f \) over these rectangles by taking test points. The only difference will be the area form that we pick up.

Let’s describe the above polar rectangle in more general terms. Suppose the circle of radius \( r_0 \) and the ray of angle \( \theta_0 \) both run through the center of the rectangle, so that the region is given by

\[ \mathcal{R} = \left\{ (r, \theta) : r_0 - \frac{1}{2}\Delta r \leq r \leq r_0 + \frac{1}{2}\Delta r, \theta_0 - \frac{1}{2}\Delta \theta \leq \theta \leq \theta_0 + \frac{1}{2}\Delta \theta \right\}. \]

We can find the area of the polar rectangle by first finding the area of the annulus in which the rectangle is contained. This area is the difference of the area of two circles:

\[
\text{Annulus area} = \pi (r_0 + \frac{1}{2}\Delta r)^2 - \pi (r_0 - \frac{1}{2}\Delta r)^2
= \pi \left[ r_0^2 + r_0\Delta r + \frac{1}{4}(\Delta r)^2 - (r_0^2 - r_0\Delta r + \frac{1}{4}(\Delta r)^2) \right]
= \pi [2r_0\Delta r] = 2\pi r_0\Delta r.
\]

The area of our wedge is then some fraction of the area of the annulus. In particular, it’s given by multiplying the annulus’s area by \( \Delta \theta/2\pi \):

\[
\text{Wedge area} = (2\pi r_0\Delta r) \frac{\Delta \theta}{2\pi} = r_0\Delta r \Delta \theta.
\]
So making small changes in angle and radius — represented by $\Delta r$ and $\Delta \theta$, respectively — results in a small change in area given by $\Delta A = r_0 \Delta r \Delta \theta$. On an infinitesimal level this takes the form $dA = rdrd\theta$. We’ll spare ourselves the run through the standard Riemann sum argument and conclude the following theorem:

**Theorem 1.** Suppose $f$ is integrable on the polar rectangle 

$$\mathcal{R} = \{(r, \theta) : r_0 \leq r \leq r_1, \theta_0 \leq \theta \leq \theta_1\}.$$ 

Then 

$$\int \int_{\mathcal{R}} f(r, \theta)dA = \int_{\theta_0}^{\theta_1} \int_{r_0}^{r_1} f(r, \theta)rdrd\theta = \int_{r_0}^{r_1} \int_{\theta_0}^{\theta_1} f(r, \theta)rdrd\theta.$$ 

(Notice that our area form is now $rdrd\theta$; a common mistake when computing double integrals in polar coordinates is to forget the $r$.)

Of course, polar rectangles aren’t the only regions we can describe with polar coordinates, and the next theorem gives us a way to compute integrals over some more general regions.

**Theorem 2.** Suppose $f$ is integrable on the polar region 

$$\mathcal{R} = \{(r, \theta) : r_0(\theta) \leq r \leq r_1(\theta), \theta_0 \leq \theta \leq \theta_1\}.$$ 

Then 

$$\int \int_{\mathcal{R}} f(r, \theta)dA = \int_{\theta_0}^{\theta_1} \int_{r_0(\theta)}^{r_1(\theta)} f(r, \theta)rdrd\theta,$$

**Example.** Let $\mathcal{R}$ be the region in $\mathbb{R}^2$ which lies outside the circle of radius 2 and inside the cardioid $r = 2(1 + \cos \theta)$, as seen below:

Compute $\int \int_{\mathcal{R}} \sin \theta dA$. 
(Solution) Since each point of this region lies in the first quadrant, the angle $\theta$ associated to each point is between 0 and $\pi/2$. So the region is given by

$$R = \{(r, \theta) : 2 \leq r \leq 2(1 + \cos \theta), 0 \leq \theta \leq \frac{\pi}{2}\}.$$ 

Then we have

$$\int\int_{R} \sin \theta dA = \int_{0}^{\pi/2} \int_{2}^{2(1+\cos \theta)} \sin \theta r dr d\theta = \int_{0}^{\pi/2} \sin \theta \left[ \frac{r^2}{2} \right]_{2}^{2(1+\cos \theta)} d\theta$$

$$= \int_{0}^{\pi/2} 2\sin \theta ((1 + \cos \theta)^2 - 1) d\theta = \int_{0}^{\pi/2} 2\sin \theta (2\cos \theta + \cos^2 \theta) d\theta.$$ 

From here we can make the substitution $u = \cos \theta$. Then $du = -\sin \theta d\theta$ and while $\theta$ varies from 0 to $\pi/2$, $u$ varies from 1 to 0. So our integral becomes

$$\int\int_{R} \sin \theta dA = -2 \int_{0}^{1} (2u + u^2) du = 2 \int_{0}^{1} (2u + u^2) du$$

$$= 2 \left[ u^2 + \frac{u^3}{3} \right]_{0}^{1} = 2 \left( 1 + \frac{1}{3} \right) = \frac{8}{3}.$$ 

Finally, here’s an example that illustrates the need for polar integration. We have an integral that would be nearly impossible to compute in rectangular coordinates, but is made much easier by changing to rectangular coordinates.

**Example. (Bivariate standard normal distribution.)** Suppose we have two independent random variables $x$ and $y$ which follow a standard normal distribution. Among other things, this means that $x$ and $y$ both have mean 0 and standard deviation 1, and the variables have no covariance – they don’t affect each other. The joint probability distribution function of these variables is given by

$$f(x, y) = \frac{1}{2\pi} \exp \left( -\frac{1}{2} (x^2 + y^2) \right).$$

This means that integrating $f$ over a region in the plane gives the probability that $(x, y)$ lies in this region. For example, if we want to know the probability that $x$ and $y$ both exceed 2, we integrate $f$ over

$$D := \{(x, y) : x \geq 2, y \geq 2\}$$

and find

$$P(x \geq 2, y \geq 2) = \int\int_{D} f(x, y) dA \approx 0.0005.$$ 

One of the properties that a probability density function must have is that its integral over its entire domain is 1. That is, the probability that $(x, y)$ lies somewhere in $\mathbb{R}^2$ must be 1. Verify that $f$ has this property.
(Solution) We’d like to show that

\[
1 = \iint_{\mathbb{R}^2} f(x, y) \, dA = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{2\pi} \exp \left( -\frac{1}{2} (x^2 + y^2) \right) \, dx \, dy.
\]

In rectangular coordinates, there’s no obvious way to proceed with the integral on the right. Thankfully, we can now transform this into an integral in polar coordinates. Since \(x^2 + y^2 = r^2\) and \(dxdy = rdrd\theta\), our integral becomes

\[
\iint_{\mathbb{R}^2} f(x, y) \, dA = \int_{\theta_0}^{\theta_1} \int_{r_0}^{r_1} \frac{1}{2\pi} \exp \left( -\frac{1}{2} r^2 \right) \, r \, dr \, d\theta,
\]

where the bounds of integration remain to be determined. Since we can describe the plane \(\mathbb{R}^2\) in polar coordinates by

\[
\mathbb{R}^2 = \{(r, \theta) : 0 \leq r < \infty, 0 \leq \theta < 2\pi\},
\]

our integral is

\[
\iint_{\mathbb{R}^2} f(x, y) \, dA = \frac{1}{2\pi} \int_0^{2\pi} \int_{r_0}^{r_1} \exp \left( -\frac{1}{2} r^2 \right) \, r \, dr \, d\theta.
\]

We now make the substitution \(u = \frac{1}{2} r^2\). Then \(du = rdr\) and \(u\) varies from 0 to \(\infty\), so

\[
\iint_{\mathbb{R}^2} f(x, y) \, dA = \frac{1}{2\pi} \int_0^{2\pi} \int_0^{\infty} e^{-u} dud\theta = \frac{1}{2\pi} \int_0^{2\pi} \left[ -e^{-u} \right]_0^{\infty} \, d\theta = \frac{1}{2\pi} \cdot 2\pi = 1.
\]

So \(f\) has the desired property. Notice that we saw no way forward in rectangular coordinates, but in polar coordinates the integral was relatively easy. Also notice that this is a case where the region over which we’re integrating is not difficult to describe in rectangular coordinates, but the integrand itself presents problems in rectangular coordinates. Sometimes we pass to polar coordinates to simplify the description of our region, and sometimes we want to simplify the integrand.