This week we’ll start exactly where you’d expect a course on multivariable integral calculus to begin — by introducing integration of multivariable functions. The definition of multiple integrals is very similar to the definition of the single-variable integral, and once we have a helpful result, the evaluation of these integrals will seem familiar as well.

Integration over rectangular regions

First let’s remember how integration is defined for a single-variable function $f$. The definite integral of $f$ over the interval $[a, b]$ is defined to compute the area between the graph of $f$ and the $x$-axis as $x$ varies from $a$ to $b$, and we have

$$\int_a^b f(x)\,dx := \lim_{n \to \infty} \sum_{i=0}^{n-1} f(a + i\Delta x)\Delta x,$$  \hspace{1cm} (1)

where $\Delta x = (b - a)/n$. The ugly looking formula on the right comes from splitting the interval $[a, b]$ into $n$ equal subintervals, and then using the function’s value at the left endpoint of each subinterval to estimate the area under the curve on this subinterval. As $n$ grows without bound, the subintervals become smaller and our approximation approaches the true area under the curve.

We go about defining the definite integral of a two-variable function in a very similar way. Plotting a two-variable function yields a graph in three dimensions, such as the one below:

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1 The definition of the definite integral in (1) assumes that $f$ is integrable. To rigorously define the definite integral we’d have to introduce the notion of partitions and then determine whether or not a given function is integrable on our interval; when the function is integrable, the equation in (1) will hold.
When a single-variable function gave us a plot in two dimensions, we computed the area under our graph; now that we have a plot in three dimensions, we’ll compute the *volume* between the graph of our function and the \(xy\)-plane. We do this in the same way we computed area in the single-variable case. We’ll slice our domain up into smaller regions, then estimate the height of our function over these smaller regions in order to estimate the volume over these regions. As the sub-regions get smaller, our approximation should become more accurate. (Pages 838 and 839 of the textbook have some good illustrations of this process.) For now we’ll restrict our attention to the case where our domain is a rectangle. In this case, chopping up our domain results in many more rectangles, and the resulting approximation looks a lot like the single variable case: We define\(^2\) the **definite integral** of \(f(x, y)\) over the rectangular region

\[
\mathcal{R} = \{(x, y) : a \leq x \leq b, c \leq y \leq d\}
\]

\[
\iint_{\mathcal{R}} f(x, y) dA := \lim_{n \to \infty} \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} f(a + i\Delta x, c + j\Delta y) \Delta x \Delta y,
\]

\[(2)\]

where \(\Delta x = (b - a)/n\) and \(\Delta y = (d - c)/n\). The definite integral over two-dimensional regions enjoys many of the nice properties we have for the single-variable integral, including **linearity**: given two integrable functions \(f(x, y)\) and \(g(x, y)\) on a region \(\mathcal{R}\) and two constants \(C, K \in \mathbb{R}\),

\[
\iint_{\mathcal{R}} (Cf(x, y) + Kg(x, y)) dA = C \iint_{\mathcal{R}} f(x, y) dA + K \iint_{\mathcal{R}} g(x, y) dA.
\]

And just as integrating the constant function 1 over an interval results in the interval’s length, integrating 1 over a rectangular region results in the region’s area:

\[
\iint_{\mathcal{R}} 1 dA = \text{Area}(\mathcal{R}).
\]

Unfortunately, the two-variable integral shares another characteristic with its single-variable cousin: computing it from the definition is at best unpleasant, and at worst impossible (remember Riemann sums?). Thankfully, **just as in the single-variable case**, we have an easier way to compute the two-variable integral. This time our savior is **Fubini’s theorem**.

**Theorem 1** (Fubini). If \(f(x, y)\) is integrable over the rectangular region

\[
\mathcal{R} = \{(x, y) : a \leq x \leq b, c \leq y \leq d\}
\]

then

\[
\iint_{\mathcal{R}} f(x, y) dA = \int_{a}^{b} \left( \int_{c}^{d} f(x, y) dy \right) dx = \int_{c}^{d} \left( \int_{a}^{b} f(x, y) dx \right) dy.
\]

\(^2\)As with the single variable case, this definition assumes that our function is integrable, which isn’t always the case.
This is great news! Computing integrals over regions using the definition would be a messy, difficult task; Fubini’s theorem says we can compute these integrals as iterated integrals — that is, we can integrate with respect to one variable and then the other, and the order in which we do this integration doesn’t matter. Single-variable integrals can be pretty messy, but they’re surely better than using the formula in (2).

Example. To demonstrate a use of Fubini’s theorem, let’s return to the function whose plot appeared earlier, which is \( f(x, y) = 20 - x^2 - y^2 \) defined on the region

\[
\mathcal{R} = \{(x, y) : -2 \leq x \leq 2, -2 \leq y \leq 2\}.
\]

Fubini’s theorem says that the volume of the region between the graph of \( f \) and the \( xy \)-plane is given by

\[
\int_{-2}^{2} \left( \int_{-2}^{2} (20 - x^2 - y^2) \, dx \right) \, dy = \int_{-2}^{2} \left( \left[ \frac{20x - x^3}{3} - xy^2 \right]_{-2}^{2} \right) \, dy
\]

\[
= \int_{-2}^{2} \left( \left[ 40 - \frac{8}{3} - 2y^2 \right] - \left[ -40 + \frac{8}{3} + 2y^2 \right] \right) \, dy
\]

\[
= \int_{-2}^{2} \left( 80 - \frac{16}{3} - 4y^2 \right) \, dy = \left[ 80y - \frac{16}{3}y - \frac{4}{3}y^3 \right]_{-2}^{2} = \frac{832}{3}
\]

Notice that we chose to integrate with respect to \( x \) first, and then \( y \). Fubini’s theorem says we could have reversed the order of integration with no effect on the end result, but sometimes the integral is much easier to compute using one order than it is using the other, as in the following example.

Example. Let \( \mathcal{R} = \{(x, y) : 0 \leq x \leq 1, 0 \leq y \leq 1\} \) and compute

\[
\int_{\mathcal{R}} y\sqrt{1+xy} \, dA.
\]

(Solution) If we integrate with respect to \( y \) first, we have

\[
\int_{\mathcal{R}} y\sqrt{1+xy} \, dA = \int_{0}^{1} \left( \int_{0}^{1} y\sqrt{1+xy} \, dy \right) \, dx.
\]

But this requires us to find an antiderivative of \( y\sqrt{1+xy} \) as a function of \( y \). That is, we want a function \( F(x, y) \) so that \( \frac{\partial F}{\partial y} = y\sqrt{1+xy} \). In this case, it’s much easier to compute

\[
\int_{\mathcal{R}} y\sqrt{1+xy} \, dA = \int_{0}^{1} \left( \int_{0}^{1} y\sqrt{1+xy} \, dx \right) \, dy,
\]

because we can let \( u = 1 + xy \). Then \( du = ydx \) (treating \( y \) as a constant) and our bounds transform to \( u = 1 \) and \( u = 1 + y \). So

\[
\int_{\mathcal{R}} y\sqrt{1+xy} \, dA = \int_{0}^{1} \left( \int_{0}^{1} y\sqrt{1+xy} \, dx \right) \, dy = \int_{0}^{1} \left( \int_{1}^{1+y} \sqrt{u} \, du \right) \, dy
\]

3
\[= \int_0^1 \left[ \frac{2}{3} u^{3/2} \right]^{1+y}_1 \, dy = \frac{2}{3} \int_0^1 \left( (1 + y)^{3/2} - 1 \right) \, dy\]
\[= \frac{2}{3} \left[ \frac{2}{5} (1 + y)^{5/2} - 1 \right]_0^1 = \frac{2}{3} \left( \frac{2}{5} (2^{5/2} - 1) - 1 \right).\]

The result still isn’t a very pretty number, but by integrating with respect to \(x\) first, we can at least evaluate the integral.

**Integration over more general regions**

We can now compute the volume under the graph of a function which is defined on a rectangle \(R\) in \(\mathbb{R}^2\), but what about more general regions? For example, consider the cylinder given below:

This is the region below the graph of \(f(x, y) = 6\) on the region
\[\mathcal{D} = \{(x, y) : x^2 + y^2 \leq 1 \} \subset \mathbb{R}^2.\]

We would like to compute this volume in the same way we compute the volume under a graph over a rectangular region, but it’s a lot trickier to write down a nice Riemann sum for this. Our solution is to consider a rectangle that includes our region, and integrate the function \(\tilde{f}\) which agrees with \(f\) on \(\mathcal{D}\) and vanishes on \(R\). We then define the integral of \(f\) over \(\mathcal{D}\) to be the integral of \(\tilde{f}\) over \(R\).

**Definition.** Let \(f(x, y)\) be defined on a bounded region \(\mathcal{D}\) and let \(R\) be a bounded rectangle containing \(\mathcal{D}\). We set
\[\tilde{f}(x, y) := \begin{cases} f(x, y), & (x, y) \in \mathcal{D} \\ 0, & (x, y) \not\in \mathcal{D} \end{cases}\]
and then define
\[\iint_{\mathcal{D}} f(x, y) dA := \iint_R \tilde{f}(x, y) dA,\]
assuming the integral on the right exists.
This new integral, defined on arbitrary\textsuperscript{3} regions, shares the nice linearity and area properties mentioned above, but we don’t immediately have a nice way to evaluate such integrals. To make this a little easier, let’s first restrict our attention to simpler regions.

We say that a region is \textbf{vertically simple} if it is bounded on the left and right by vertical lines, and above and below by continuous functions \( y = g_1(x) \) and \( y = g_2(x) \). For example, the region on the left is vertically simple:

![Diagram of a vertically simple region]

Analogously, we call a region \textbf{horizontally simple} if it is bounded above and below by horizontal lines and bounded on the left and right by continuous functions \( x = g_1(y) \) and \( x = g_2(y) \). See the region on the right above. We can describe vertically simple and horizontally simple regions as sets by

\[
D_V = \{ (x, y) : a \leq x \leq b, g_1(x) \leq y \leq g_2(x) \}
\]

and

\[
D_H = \{ (x, y) : g_1(y) \leq x \leq g_2(y), c \leq y \leq d \},
\]

respectively. The advantage these regions have is that we can make slices which are parallel to the straight sides of the region. That is, for a vertically simple region, imagine fixing an \( x \)-value between \( a \) and \( b \) and then integrating \( f(x, y) \) with respect to \( y \) from the lower bound of our region to the upper bound. For a fixed \( x \)-value, this gives

\[
\int_{g_1(x)}^{g_2(x)} f(x, y) dy.
\]

We can then integrate this value with respect to \( x \) over the interval \([a, b]\), and the result should be \( \iint_{D_V} f(x, y) dA \); in fact, it is.

\textbf{Theorem 2.} If \( f(x, y) \) is integrable on the vertically simple region

\[
D_V = \{ (x, y) : a \leq x \leq b, g_1(x) \leq y \leq g_2(x) \},
\]

\textsuperscript{3}We’re pretending that double integrals can be defined on any region we like, but there are actually some strange, technical properties that a region (and the functions defined on it) must satisfy in order for these integrals to make sense. Those who are interested can study this problem as part of an area called \textit{measure theory}.
then
\[
\int\int_{D_V} f(x, y) \, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y) \, dy \, dx.
\]

Similarly, if \( f(x, y) \) is integrable on the horizontally simple region
\[
D_H = \{(x, y) : g_1(y) \leq x \leq g_2(y), c \leq y \leq d\},
\]
then
\[
\int\int_{D_H} f(x, y) \, dA = \int_c^d \int_{g_1(y)}^{g_2(y)} f(x, y) \, dx \, dy.
\]

**Example.** Let \( D \) be the region in the plane bounded on the left and right by \( x = -1 \) and \( x = 1 \), and between \( y = x^2 - 2 \) and \( y = 2 - x^2 \).

Let \( f(x, y) = x + y + 3 \) and compute
\[
\int\int_D f(x, y) \, dA.
\]

On the left is the region \( D \), and on the right is the plot of \( f \) over \( D \).

*(Solution)* Since the region \( D \) is vertically simple, we have
\[
\int\int_D f(x, y) \, dA = \int_{-1}^1 \int_{x^2-2}^{2-x^2} x + y + 3 \, dy \, dx = \int_{-1}^1 \left[ (x + 3)y + \frac{y^2}{2} \right]_{x^2-2}^{2-x^2} \, dx
\]
\[
= \int_{-1}^1 \left[ \left( x + 3 \right) \left( 2 - x^2 \right) + \frac{2 - x^2}{2} \right] - \left( x + 3 \right) \left( x^2 - 2 \right) + \frac{\left( x^2 - 2 \right)^2}{2} \right] \, dx
\]
\[
= \int_{-1}^1 2(x + 3)(2 - x^2) \, dx = 2 \int_{-1}^1 \left( 6 + 2x - 3x^2 - x^3 \right) \, dx
\]
\[
= 2 \left[ 6x + x^2 - x^3 - \frac{x^4}{4} \right]_{-1}^1 = 2 \left( 6 + 1 - 1 - \frac{1}{4} \right) - 2 \left( -6 + 1 + 1 - \frac{1}{4} \right)
\]
\[
= 20.
\]
**Example.** Let’s return to the cylinder we plotted at the beginning of this section. This was the region below the graph of \( f(x, y) = 6 \) over the unit disk \( \mathcal{D} \) in the plane. At first glance the unit disk doesn’t seem to be either vertically or horizontally simple, but in fact it’s both. Notice that we can write

\[
\mathcal{D} = \{ (x, y) : x^2 + y^2 \leq 1 \} = \{ (x, y) : -1 \leq x \leq 1, \ -\sqrt{1-x^2} \leq y \leq \sqrt{1-x^2} \}
\]

to see that \( \mathcal{D} \) is vertically simple, or write

\[
\mathcal{D} = \{ (x, y) : -1 \leq x \leq 1, \ -\sqrt{1-y^2} \leq x \leq \sqrt{1-y^2} \}
\]

to see that \( \mathcal{D} \) is also horizontally simple. This is important, since we currently only know how to integrate over regions which are simple in one of these senses. Using the second part of Theorem 2, we have

\[
\iint_{\mathcal{D}} f(x, y) dA = \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} 6 dy dx = \int_{-1}^{1} [6x] \sqrt{1-y^2} dy = \int_{-1}^{1} \left( 6\sqrt{1-x^2} - 6\left(-\sqrt{1-y^2}\right) \right) dy = 12 \int_{-1}^{1} \sqrt{1-y^2} dy.
\]

At this point we make the change of variables \( y = \sin \theta \). Then \( dy = \cos \theta d\theta \) and our bounds become \(-\pi/2\) and \(\pi/2\). So

\[
\iint_{\mathcal{D}} f(x, y) dA = 12 \int_{-\pi/2}^{\pi/2} \cos^2 \theta d\theta d\theta = 12 \int_{-\pi/2}^{\pi/2} \frac{1 + \cos(2\theta)}{2} d\theta = 12 \left[ \frac{\theta + \frac{1}{2} \sin(2\theta)}{2} \right]_{-\pi/2}^{\pi/2} = 6 \left( \frac{\pi}{2} + \frac{0}{2} \right) - 6 \left( -\frac{\pi}{2} + \frac{0}{2} \right) = 6\pi.
\]

For this particular example we have a nice reality check: we know how to compute the volume of a right circular cylinder by knowing its height and radius, and this volume should agree with the volume we computed by integration.