We’ve recently spent a lot of time concerning ourselves with different coordinate systems and how we can use them to compute integrals. An important thing to track when making these changes is the integrating factor that appears. For instance, integrating in polar coordinates slipped an $r$ into our area form:

$$\iint_{D_f} f(x, y) \, dx \, dy = \iint_{D_0} f(r \cos \theta, r \sin \theta) r \, dr \, d\theta.$$  \hfill (1)

In these notes we want to compute the integrating factor for an arbitrary change of variables, greatly expanding the number of coordinate systems we can place on $\mathbb{R}^2$ and $\mathbb{R}^3$, thus giving us more firepower in computing integrals.

Note. We won’t actually cover most of the theoretical contents of these notes in section. I’m just posting them here for those who are curious about where our change of variables formulas come from.

**The Jacobian Determinant in Two Variables**

When we define a change of coordinates on $\mathbb{R}^2$, we usually write it as

$$x = x(u, v), \quad y = y(u, v),$$

where $x(u, v)$ and $y(u, v)$ are some nice functions of two variables. For instance, we’ve grown fond of the coordinate change

$$x = r \cos \theta, \quad y = r \sin \theta.$$  

We can then define integration in this new coordinate system by subdividing the $xy$-plane into small, simple regions, as we did for rectangular coordinates, and as we pretended to do for polar coordinates. We’ve seen, however, that the resulting regions will not be rectangles — they’ll be rectangles in our new coordinate system, whatever that means. The following plots show how our rectangular system gets transformed into a much stranger system:

On the left we have a rectangle in the $uv$-plane, and this rectangle corresponds to the strange blue shape in the $xy$-plane on the right. Whatever this shape is, we need to approximate its area in terms of $u$ and $v$. Let’s suppose that the rectangle in the $uv$-plane has side lengths $\Delta u$ and
\[ \Delta v, \text{ and we’ll use } \Delta A \text{ to denote the area of the blue region in the } xy\text{-plane. We see that this region is not quite a parallelogram, but perhaps we can approximate it by one, as shown in the figure.} \]

The most natural parallelogram to approximate our region with is probably the one whose four corners are the four vertices of the blue region. If the bottom vertex corresponds to the inputs \((u_0, v_0)\), then we want the parallelogram with vertices
\[
(x(u_0, v_0), y(u_0, v_0)), \quad (x(u_0 + \Delta u, v_0), y(u_0 + \Delta u, v_0)), \quad (x(u_0, v_0 + \Delta v), y(u_0, v_0 + \Delta v)), \quad (x(u_0 + \Delta u, v_0 + \Delta v), y(u_0 + \Delta u, v_0 + \Delta v)).
\]

Let’s consider the vector along the bottom right of this parallelogram; that is, we’re interested in the parallelogram connecting \(T(u_0, v_0)\) to \(T(u_0 + \Delta u, v_0)\), where \(T(u, v) = (x(u, v), y(u, v))\) transforms the \(uv\)-plane into the \(xy\)-plane. So we want the vector
\[
T(u_0 + \Delta u, v_0) - T(u_0, v_0).
\]

Since only the \(u\)-input is changing, we can use a local linear approximation in the \(u\)-variable to approximate this vector:
\[
T(u_0 + \Delta u, v_0) - T(u_0, v_0) \approx \Delta u \frac{\partial T}{\partial u}(u_0, v_0),
\]

where by \(\frac{\partial T}{\partial u}\) we mean the vector
\[
\frac{\partial T}{\partial u} = \left\langle \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u} \right\rangle.
\]

Indeed, the parallelogram shown in the above figure uses \(\Delta u \frac{\partial T}{\partial u}\) as its bottom-right vector.

We can similarly approximate the bottom-left of our region by the vector \(\Delta v \frac{\partial T}{\partial v}\). Altogether, we’re approximating our region with the parallelogram determined by
\[
\Delta u \left\langle \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u} \right\rangle \quad \text{and} \quad \Delta v \left\langle \frac{\partial x}{\partial v}, \frac{\partial y}{\partial v} \right\rangle,
\]

where each of these partial derivatives is evaluated at the point \((u_0, v_0)\). Now the area of a parallelogram formed from vectors \(a\) and \(b\) is the (absolute value of the) determinant of the matrix made from these vectors. In our case, the pink parallelogram has area given by the absolute value of
\[
\Delta u \Delta v = \left| \begin{array}{cc} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{array} \right| \Delta u \Delta v.
\]

We call the expression in parentheses in (2) the **Jacobian determinant** of our transformation, and we often denote it by \(\dfrac{\partial(x,y)}{\partial(u,v)}\). You may also recall that the area of the parallelogram spanned by two vectors is the magnitude of the cross product of these two vectors, and indeed, if we think of our vectors as living in three dimensions and compute the area this way, we will find the same expression given in (2). Our approximation then tells us that
\[
\Delta A \approx \frac{\partial(x,y)}{\partial(u,v)} \Delta u \Delta v.
\]
While this approximation isn’t great for large values of $\Delta u$ and $\Delta v$, the following theorem tells us that it’s a very good approximation as $\Delta u, \Delta v \to 0$.

**Theorem 1** (Change of Variables Formula). Let $G: D_0 \to D$ be a continuously differentiable function which is injective on the interior of $D_0$, where $D$ and $D_0$ are regions in $\mathbb{R}^2$. If $f(x,y)$ is continuous on $D$, then

$$
\int\int_D f(x,y) \, dx \, dy = \int\int_{D_0} f(x(u,v),y(u,v)) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| \, du \, dv,
$$

where $G(u,v) = (x(u,v),y(u,v))$.

We now know how our change of variables will impact our integration factor — we’ll multiply by the absolute value of the determinant of the Jacobian matrix.

**Example.** Use Theorem 1 to verify that the equation in (1) is correct.

*(Solution)* For (1) we were using the change of variables given by polar coordinates:

$$
x = x(r,\theta) = r \cos \theta, \quad y = y(r,\theta) = r \sin \theta.
$$

Then our Jacobian matrix is given by

$$
\begin{pmatrix}
x_r & x_\theta \\
y_r & y_\theta
\end{pmatrix} = \begin{pmatrix}
\cos \theta & -r \sin \theta \\
\sin \theta & r \cos \theta
\end{pmatrix},
$$

so our Jacobian determinant is

$$
\frac{\partial(x,y)}{\partial(r,\theta)} = r \cos^2 \theta - (-r \sin^2 \theta) = r(\cos^2 \theta + \sin^2 \theta) = r.
$$

From Theorem 1 we see that

$$
\int\int_D f(x,y) \, dx \, dy = \int\int_{D_0} f(r \cos \theta, r \sin \theta) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| \, dr \, d\theta = \int\int_{D_0} f(r \cos \theta, r \sin \theta) r \, dr \, d\theta,
$$

as desired. ♦

**Example.** Use the map $T(u,v) = \left( \frac{u}{v+1}, \frac{uv}{v+1} \right)$ to compute

$$
\int\int_D (x + y) \, dx \, dy,
$$

where $D$ is the shaded region below:
There’s a lot going on in this problem. First let’s describe $D$ as a set. We have

$$D = \{(x, y) : ax \leq y \leq bx, c \leq x + y \leq d\} = \{(x, y) : a \leq \frac{y}{x} \leq b, c \leq x + y \leq d\}.$$ 

We can divide our first inequality by $x$ to obtain the second set description of $D$ because we have $x > 0$. Now since

$$x(u, v) = \frac{u}{v + 1}, \quad y(u, v) = \frac{uv}{v + 1},$$

we see that

$$\frac{y}{x} = \frac{uv}{v + 1} = v \quad \text{and} \quad x + y = \frac{u + uv}{v + 1} = \frac{u(1 + v)}{v + 1} = u.$$ 

So if we set

$$D_0 = \{(u, v) : a \leq v \leq b, c \leq u \leq d\},$$

then $G(D_0) = D$. Next, let’s compute our Jacobian determinant. Notice that our partials are

$$\frac{\partial x}{\partial u} = \frac{1}{v + 1}, \quad \frac{\partial x}{\partial v} = \frac{-u}{(v + 1)^2},$$

$$\frac{\partial y}{\partial u} = \frac{v}{v + 1}, \quad \frac{\partial y}{\partial v} = \frac{u(v + 1) - uv}{(v + 1)^2} = \frac{u}{(v + 1)^2}.$$ 

Then our Jacobian determinant is given by

$$\frac{\partial(x, y)}{\partial(u, v)} = \frac{\partial x \partial y}{\partial u \partial v} - \frac{\partial x \partial y}{\partial v \partial u} = \frac{u}{(v + 1)^3} - \frac{-uv}{(v + 1)^3} = \frac{u(v + 1)}{(v + 1)^3} = \frac{u}{(v + 1)^2}.$$ 

Since $u, v > 0$ in our region we have

$$\left| \frac{\partial(x, y)}{\partial(u, v)} \right| = \frac{u}{(v + 1)^2}.$$ 

Finally, we can use Theorem 1 to conclude that

$$\iint_D (x + y) \, dxdy = \iint_{D_0} u \left| \frac{\partial(x, y)}{\partial(u, v)} \right| \, dudv = \int_a^b \int_c^d \frac{u^2}{(v + 1)^2} dvdu.$$
\[
\int_a^b \left( \frac{u^3}{3(v+1)^2} \right)_d \, dv = \int_a^b \frac{d^3 - c^3}{3(v+1)^2} \, dv \\
= \left[ \frac{-(d^3 - c^3)}{3(v+1)} \right]_a^b \\
= \frac{c^3 - d^3}{3(b+1)} - \frac{c^3 - d^3}{3(a+1)}.
\]

Notice that without Theorem 1 we don’t immediately have a way to write down the integral as an iterated integral.

\[\diamondsuit\]

The Jacobian Determinant in Three Variables

In addition to defining changes of coordinates on \( \mathbb{R}^3 \), we’ve defined a couple of new coordinate systems on \( \mathbb{R}^3 \) — namely, cylindrical and spherical coordinate systems. For spherical coordinates we write

\[
x = x(\rho, \theta, \phi) = \rho \cos \theta \sin \phi, \quad y = y(\rho, \theta, \phi) = \rho \sin \theta \sin \phi, \quad z = z(\rho, \theta, \phi) = \rho \cos \phi,
\]

and proceed to define integration with respect to this new coordinate system. As before, we can generalize this to the situation where we have some change of coordinates

\[
x = x(u, v, w), \quad y = y(u, v, w), \quad z = z(u, v, w).
\]

In this setup, instead of rectangles being transformed into strange shapes in \( \mathbb{R}^2 \), as in the first figures of these notes, we have rectangular prisms being transformed into strange shapes in \( \mathbb{R}^3 \). Where we used a parallelogram to approximate our new area in two dimensions, we’ll use a parallelepiped to approximate our new volume in three dimensions. Recall that the parallelogram had side vectors

\[
a = \begin{pmatrix} x_u \\ y_u \\ z_u \end{pmatrix} \Delta u, \quad b = \begin{pmatrix} x_v \\ y_v \\ z_v \end{pmatrix} \Delta v.
\]

In a similar manner, the approximating parallelepiped we use in three dimensions has constituent vectors

\[
a = \begin{pmatrix} x_u \\ y_u \\ z_u \end{pmatrix} \Delta u, \quad b = \begin{pmatrix} x_v \\ y_v \\ z_v \end{pmatrix} \Delta v, \quad c = \begin{pmatrix} x_w \\ y_w \\ z_w \end{pmatrix} \Delta w.
\]

We can now find the volume of our parallelepiped using the triple scalar product:

\[
a \cdot (b \times c) = \begin{vmatrix} x_u & y_u & z_u \\ x_v & y_v & z_v \\ x_w & y_w & z_w \end{vmatrix} \Delta u \Delta v \Delta w
\]

\[
= \begin{vmatrix} x_u & y_u & z_u \\ x_v & y_v & z_v \\ x_w & y_w & z_w \end{vmatrix} \Delta u \Delta v \Delta w.
\]

Our volume form, \( \Delta V \), will then be approximated by the absolute value of the above expression. So as in the two-variable case, our volume form is not simply the product of our dimensional changes.
Also as in the two-variable case, we assign a name to the factor by which we must scale the product of our dimensional changes. We write

\[
\frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} x_u & y_u & z_u \\ x_v & y_v & z_v \\ x_w & y_w & z_w \end{vmatrix}
\]

and call this the **Jacobian determinant** of our change of variables. So we have

\[
\Delta V \approx \left| \frac{\partial(x, y, z)}{\partial(u, v, w)} \right| \Delta u \Delta v \Delta w,
\]

just as in the two-variable case. Finally, we get another change of variables formula.

**Theorem 2** (Change of Variables Formula). Let \( G: D_0 \to D \) be a continuously differentiable function which is injective on the interior of \( D_0 \), where \( D \) and \( D_0 \) are regions in \( \mathbb{R}^3 \). If \( f(x, y, z) \) is continuous on \( D \), then

\[
\iiint_D f(x, y, z) \, dx \, dy \, dz = \iiint_{D_0} f(x(u, v, w), y(u, v, w), z(u, v, w)) \left| \frac{\partial(x, y, z)}{\partial(u, v, w)} \right| \, du \, dv \, dw,
\]

where \( G(u, v, w) = (x(u, v, w), y(u, v, w), z(u, v, w)) \).
Group Work

1. Use the transformation \( u = \frac{1}{2}(x + y), v = \frac{1}{2}(x - y) \) to evaluate
\[
\iint_{\mathcal{R}} \sin\left(\frac{x + y}{2}\right) \cos\left(\frac{x - y}{2}\right) \, dA,
\]
where \( \mathcal{R} \) is the triangular region with vertices \((0, 0), (2, 0), \) and \((1, 1)\).

2. Evaluate the integrals
\[
\iint_{\mathcal{D}} (x^2 + y^2) \, dA \quad \text{and} \quad \iint_{\mathcal{D}} (x^4 - y^4) \, dA,
\]
where
\[
\mathcal{D} = \{(x, y) \in \mathbb{R}^2 \mid -3 \leq x^2 - y^2 \leq 3, 1 \leq xy \leq 4, x \geq 0\}.
\]
Hint: Consider the map \( F(x, y) = (x^2 - y^2, xy) \) applied to \( \mathcal{D} \).

Answers

1. \( 1 - \frac{1}{2} \sin(2) \)
2. 9 and 0