Today we’ll start exactly where you’d expect a course on multivariable integral calculus to begin — by introducing integration of multivariable functions. The definition of multiple integrals is very similar to the definition of the single-variable integral, and once we have a helpful result, the evaluation of these integrals will seem familiar as well.

**Integration over rectangular regions**

First let’s remember how integration is defined for a single-variable function $f$. The definite integral of $f$ over the interval $[a, b]$ is defined to compute the area between the graph of $f$ and the $x$-axis as $x$ varies from $a$ to $b$, and we have\(^1\)

\[
\int_a^b f(x)\,dx := \lim_{n\to\infty} \sum_{i=0}^{n-1} f(a + i\Delta x)\Delta x, \tag{1}
\]

where $\Delta x = (b - a)/n$. The ugly looking formula on the right comes from splitting the interval $[a, b]$ into $n$ equal subintervals, and then using the function’s value at the left endpoint of each subinterval to estimate the area under the curve on this subinterval. As $n$ grows without bound, the subintervals become smaller and our approximation approaches the true area under the curve.

We go about defining the definite integral of a two-variable function in a very similar way. Plotting a two-variable function yields a graph in three dimensions, such as the one below:

![Graph of a two-variable function](image)

When a single-variable function gave us a plot in two dimensions, we computed the area under our graph; now that we have a plot in three dimensions, we’ll compute the *volume* between the graph of our function and the $xy$-plane. We do this in the same way we computed area in the single-variable case. We’ll slice our domain up into smaller regions, then estimate the height of our function over

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\(^1\)The definition of the definite integral in (1) assumes that $f$ is integrable. To rigorously define the definite integral we’d have to introduce the notion of *partitions* and then determine whether or not a given function is integrable on our interval; when the function is integrable, the equation in (1) will hold.
these smaller regions in order to estimate the volume over these regions. As the sub-regions get smaller, our approximation should become more accurate. (Pages 838 and 839 of the textbook have some good illustrations of this process.) For now we’ll restrict our attention to the case where our domain is a rectangle. In this case, chopping up our domain results in many more rectangles, and the resulting approximation looks a lot like the single variable case: We define the definite integral of $f(x, y)$ over the rectangular region

$$
\mathcal{R} = \{(x, y) : a \leq x \leq b, c \leq y \leq d\}
$$

to be

$$
\iint_{\mathcal{R}} f(x, y) dA := \lim_{n \to \infty} \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} f(a + i\Delta x, c + j\Delta y) \Delta x \Delta y,
$$

(2)

where $\Delta x = (b - a)/n$ and $\Delta y = (d - c)/n$. The definite integral over two-dimensional regions enjoys many of the nice properties we have for the single-variable integral, including linearity: given two integrable functions $f(x, y)$ and $g(x, y)$ on a region $\mathcal{R}$ and two constants $C, K \in \mathbb{R}$,

$$
\iint_{\mathcal{R}} (Cf(x, y) + Kg(x, y)) dA = C \iint_{\mathcal{R}} f(x, y) dA + K \iint_{\mathcal{R}} g(x, y) dA.
$$

And just as integrating the constant function 1 over an interval results in the interval’s length, integrating 1 over a rectangular region results in the region’s area:

$$
\iint_{\mathcal{R}} 1 dA = \text{Area}(\mathcal{R}).
$$

Unfortunately, the two-variable integral shares another characteristic with its single-variable cousin: computing it from the definition is at best unpleasant, and at worst impossible (remember Riemann sums?). Thankfully, just as in the single-variable case, we have an easier way to compute the two-variable integral. This time our savior is Fubini’s theorem.

**Theorem 1 (Fubini).** If $f(x, y)$ is integrable over the rectangular region

$$
\mathcal{R} = \{(x, y) : a \leq x \leq b, c \leq y \leq d\}
$$

then

$$
\iint_{\mathcal{R}} f(x, y) dA = \int_a^b \left( \int_c^d f(x, y) dy \right) dx = \int_c^d \left( \int_a^b f(x, y) dx \right) dy.
$$

This is great news! Computing integrals over regions using the definition would be a messy, difficult task; Fubini’s theorem says we can compute these integrals as iterated integrals — that is, we can integrate with respect to one variable and then the other, and the order in which we do this integration doesn’t matter. Single-variable integrals can be pretty messy, but they’re surely better than using the formula in (2).

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2 As with the single variable case, this definition assumes that our function is integrable, which isn’t always the case.
Example. To demonstrate a use of Fubini’s theorem, let’s return to the function whose plot appeared earlier, which is \( f(x, y) = 20 - x^2 - y^2 \) defined on the region

\[ R = \{(x, y) : -2 \leq x \leq 2, -2 \leq y \leq 2\}. \]

Fubini’s theorem says that the volume of the region between the graph of \( f \) and the \( xy \)-plane is given by

\[
\int_{-2}^{2} \left( \int_{-2}^{2} \left( 20 - x^2 - y^2 \right) dx \right) dy = \int_{-2}^{2} \left( \left[ \frac{20x - x^3}{3} - xy^2 \right]_{-2}^{2} \right) dy
\]

\[
= \int_{-2}^{2} \left[ \left( 40 - \frac{8}{3} - 2y^2 \right) - \left( -40 + \frac{8}{3} + 2y^2 \right) \right] dy
\]

\[
= \int_{-2}^{2} \left( 80 - \frac{16}{3} - 4y^2 \right) dy = \left[ 80y - \frac{16}{3}y - \frac{4}{3}y^3 \right]_{-2}^{2} = \frac{832}{3}
\]

Notice that we chose to integrate with respect to \( x \) first, and then \( y \). Fubini’s theorem says we could have reversed the order of integration with no effect on the end result, but sometimes the integral is much easier to compute using one order than it is using the other, as in the following example.

Example. Let \( R = \{(x, y) : 0 \leq x \leq 1, 0 \leq y \leq 1\} \) and compute

\[
\int_{R} \sqrt{1 + xy} dA.
\]

(Solution) If we integrate with respect to \( y \) first, we have

\[
\int_{R} \sqrt{1 + xy} dA = \int_{0}^{1} \left( \int_{0}^{1} \sqrt{1 + xy} dy \right) dx.
\]

But this requires us to find an antiderivative of \( y\sqrt{1 + xy} \) as a function of \( y \). That is, we want a function \( F(x, y) \) so that \( \frac{\partial F}{\partial y} = y\sqrt{1 + xy} \). In this case, it’s much easier to compute

\[
\int_{R} y\sqrt{1 + xy} dA = \int_{0}^{1} \left( \int_{0}^{1} y\sqrt{1 + xy} dx \right) dy,
\]

because we can let \( u = 1 + xy \). Then \( du = ydx \) (treating \( y \) as a constant) and our bounds transform to \( u = 1 \) and \( u = 1 + y \). So

\[
\int_{R} y\sqrt{1 + xy} dA = \int_{0}^{1} \left( \int_{0}^{1} y\sqrt{1 + xy} dx \right) dy = \int_{0}^{1} \left( \int_{1}^{1+y} \sqrt{u} du \right) dy
\]

\[
= \int_{0}^{1} \left[ \frac{2}{3}u^{3/2} \right]_{1}^{1+y} dy = \frac{2}{3} \int_{0}^{1} \left( (1 + y)^{3/2} - 1 \right) dy
\]

\[
= \frac{2}{3} \left[ \frac{2}{5}(1 + y)^{5/2} - 1 \right]_{0}^{1} = \frac{2}{3} \left( \frac{2}{5}(2^{5/2} - 1) - 1 \right) = \frac{2}{15} \left( 8\sqrt{2} - 7 \right).
\]

The result still isn’t a very pretty number, but by integrating with respect to \( x \) first, we can at least evaluate the integral.
Group Work

1. Evaluate the following integrals:

   (a) \( \int_4^9 \int_{-3}^8 1 \, dx \, dy \)

   (b) \( \iint_{R} \cos x \sin 2y \, dA \), where \( R = [0, \frac{\pi}{2}] \times [0, \frac{\pi}{2}] \)

2. Evaluate \( \int_0^1 \int_0^1 \frac{y}{1 + xy} \, dy \, dx \). Hint: Change the order of integration.

Answers

1. (a) 55
   (b) 1

2. \( 2 \ln 2 - 1 \)