WEEK 9 WORKSHEET

(1) The critical points of a function $f$ are the points $(a, b)$ where $\nabla f(a, b) = 0$. A critical point is called degenerate if

$$\det \begin{bmatrix} f_{xx}(a, b) & f_{xy}(a, b) \\ f_{yx}(a, b) & f_{yy}(a, b) \end{bmatrix} = 0.$$ 

Find all critical points of $f(x, y) = 6xy^2 - 2x^3 - 3y^4$ and identify the ones which are degenerate.

(2) Consider the function $f(x, y) = x^2 + 4y^2 - 4xy + 2$.

(a) Show that $f(x, y)$ has an infinite number of critical points.

(b) Show that all of the critical points are degenerate.

(c) Verify that $f(a, b) = 2$ for every critical point $(a, b)$. Use this to show that $f$ has an absolute minimum at each critical point. Hint: factor $x^2 - 4xy + 4y^2$.

(3) Consider $f(x, y) = 2 \cos x - y^2 + e^{xy}$.

(a) Verify that $(0, 0)$ is a critical point for $f$.

(b) Calculate each of $f_{xx}, f_{xy}, f_{yy}$ at $(0, 0)$. The second-order Taylor polynomial at $(0, 0)$ is given by

$$T_2(x, y) = f(0, 0) + f_x(0, 0)x + f_y(0, 0)y + \frac{1}{2} f_{xx}(0, 0)x^2 + f_{xy}(0, 0)xy + \frac{1}{2} f_{yy}(0, 0)y^2.$$ 

What is $T_2(x, y)$ for our function $f(x, y)$ at $(0, 0)$?

(4) Let $g(x, y) = T_2(x, y)$ that you computed in 3(b). Since $g$ is much simpler than $f$, we will use it to determine whether $f$ has a minimum, a maximum, or a saddle point at $(0, 0)$.

(a) Determine the behavior of $g$ along the line $y = mx$; that is, replace $y$ by $mx$ so that $g(x, mx)$ is a single variable function. Now use what you know from calc I to find out what $g$ looks like for each value of $m$.

(b) Use your answer from (a) to make a guess about the behavior of $f$ at $(0, 0)$.

(5) Consider alternate coordinates $(u, v)$ on $\mathbb{R}^2$ given by $(x, y) = (u - v, u + v)$.

(a) Express the function $g$ from problem 4 as a function of $u$ and $v$, and expand and simplify the resulting expression.

(b) Explain why your answer in 5(a) confirms your guess in 4(b).

(c) Sketch a contour map for $g$ (in terms of $u$ and $v$) near $(0, 0)$. What does this tell you about the contour map of $f$ near $(0, 0)$?

(It turns out that there is always a similar change of coordinates so that the Taylor series of a function $f$ which has a critical point at $(0, 0)$ looks like $f(u, v) \approx f(0, 0) + au^2 + bv^2$. In fact this is why the 2nd derivative test works.)
(6) For functions of one variable it is impossible for a continuous function to have two local maxima and no local minimum (why is that?). However, for functions of two variables such functions exist. Find the critical points for
\[ f(x, y) = -(x^2 - 1)^2 - (x^2 y - x - 1)^2 \]
and show that \( f \) has a local maximum at each point.

(7) Three alleles (alternative versions of a gene) A, B, and O determine the four blood types A (AA or AO), B (BB or BO), O (OO) and AB. The Hardy-Weinberg Law states that the proportion of individuals in a population who carry two different alleles is

\[ P = 2pq + 2pr + 2rq \]

where \( p, q, r \) are, respectively, the proportions of A, B, and O in the population.

(a) Explain why \( p + q + r = 1 \).
(b) Show that \( P \) is at most \( \frac{2}{3} \).
   
   Hint: first use that \( p + q + r = 1 \) to turn \( P \) into a function of two variables.

(8) Find the maximum volume of the largest box with one corner at the origin and the opposite corner at a point \( P = (x, y, z) \) on the paraboloid

\[ z = 1 - \frac{x^2}{4} - \frac{y^2}{9} \]

with \( x, y, z \geq 0 \).