DISCUSSION 10

1. Consider the functions \( f(x, y) = xy \) and \( g(x, y) = \frac{x^2}{36} + \frac{y^2}{9} \). Here’s a plot with a few level curves of \( f \) (in blue) and the level curve \( g(x, y) = 1 \) (in red):

Our goal is to identify the point on the red curve where \( f \) is largest (subject to the conditions \( x \geq 0, y \geq 0 \)).

(a) At each of the three marked points, draw a vector pointing in the direction of \( \nabla f \).
(b) At each of the green points, determine the direction we should move along the red curve in order to increase the value of \( f \).
(c) At the blue point, what can you say about the tangent line to the red curve and the vector \( \nabla f \)? What does this allow you to conclude about \( \nabla g \) and \( \nabla f \)?
(d) Identify the coordinates of the blue point.

2. Consider an isosceles triangle inscribed in the ellipse \( \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1 \), with one vertex at the point \((0, b)\):

Find the maximum area of such a triangle. (Only the top vertex is fixed.)

3. A Cobb-Douglas utility function purports to measure the level of utility (however that might be measured) that a consumer derives from a combination of goods. Suppose that \( x \) units of product \( A \) and \( y \) units of product \( B \) give a particular consumer utility

\[
U(x, y) = 10x^{1/4}y^{3/4}.
\]

Also suppose that product \( A \) has a price of \$40, product \( B \) has a price of \$60, and our consumer has \$800 to spend on these products. How should this budget be allocated between these products to maximize utility?
4. Consider the plane in $\mathbb{R}^3$ passing through the points $(3, 0, 0)$, $(0, 3, 0)$, and $(0, 0, 3)$, with $a, b, c \neq 0$. Find the point on this plane which is nearest the origin.

5. Find the volume of the largest rectangular parallelepiped, with edges parallel to the axes, inscribed in the ellipsoid

$$\frac{x^2}{4} + \frac{y^2}{16} + \frac{z^2}{16} = 1.$$