This week we’ll spend some time thinking about limits of two-variable functions. Many of the rules and techniques we used for computing limits of single-variable functions continue to work in this setting, but a big difference has to do with proving that limits don’t exist. We’ll briefly discuss some of these differences (comments that aren’t included here), and then work a few of the examples below.

Example 1. (§15.2, Exercise 39 of the textbook) Determine whether or not the limit

$$\lim_{(x,y) \to (0,0)} \frac{x^2 + y^2}{\sqrt{x^2 + y^2 + 1} - 1}$$

exists. If the limit exists, evaluate it.

(Solution) Let’s try an approach that worked for us with single-variable limits: multiplying by a conjugate. We have

$$\lim_{(x,y) \to (0,0)} \frac{x^2 + y^2}{\sqrt{x^2 + y^2 + 1} - 1} = \lim_{(x,y) \to (0,0)} \frac{x^2 + y^2}{\sqrt{x^2 + y^2 + 1} - 1} \cdot \frac{\sqrt{x^2 + y^2 + 1} + 1}{\sqrt{x^2 + y^2 + 1} + 1}$$

$$= \lim_{(x,y) \to (0,0)} \frac{(x^2 + y^2)(\sqrt{x^2 + y^2 + 1} + 1)}{x^2 + y^2 + 1 - 1}$$

$$= \lim_{(x,y) \to (0,0)} \sqrt{x^2 + y^2 + 1} + 1 = 2.$$

The following figure shows a plot of $z = (x^2 + y^2)/\sqrt{x^2 + y^2 + 1} - 1$:

Of course $(x^2 + y^2)/\sqrt{x^2 + y^2 + 1} - 1$ is not defined at $(x, y) = (0, 0)$. Our work above, though, shows that this expression does approach a particular value (namely, 2), so we can fill the hole in our graph with the green dot you see above.

Example 2. (§15.2, Exercise 41 of the textbook) Determine whether or not the limit

$$\lim_{(x,y) \to (0,0)} \frac{x^3 + y^3}{x^2 + y^2}$$

exists. If the limit exists, evaluate it.
(Solution) We’ll show that this limit exists by applying the squeeze theorem. That is, let \( f(x, y) = \frac{x^3 + y^3}{x^2 + y^2} \); we want to find functions \( g(x, y) \) and \( h(x, y) \) so that
\[
g(x, y) \leq f(x, y) \leq h(x, y)
\]
and
\[
\lim_{(x,y) \to (0,0)} g(x, y) = L = \lim_{(x,y) \to (0,0)} h(x, y).
\]
This will allow us to conclude that the limit exists, and is \( L \). The functions we’ll use aren’t particularly obvious. We’ll start by writing
\[
f(x, y) = \frac{x^3}{x^2 + y^2} + \frac{y^3}{x^2 + y^2}.
\]
As \( x \) and \( y \) both become small, we expect the behavior of these fractions to be approximately determined by the powers of their numerators and denominators. That is, the first term should approximately behave like \( x \). Let’s make this rigorous. For any \( x \) we have
\[
|x^3| = |x||x^2| \leq |x|(x^2 + y^2).
\]
We have the equality because \( x^2 \) is a nonnegative number that we can slide out of the absolute value, and we have the inequality because adding the nonnegative number \( |x|y^2 \) will not decrease our expression. We similarly have
\[
|y^3| \leq |y|(x^2 + y^2).
\]
This means that
\[
\left| \frac{x^3}{x^2 + y^2} \right| = \frac{|x|(x^2 + y^2)}{x^2 + y^2} = |x| \quad \text{and} \quad \left| \frac{y^3}{x^2 + y^2} \right| = \frac{|y|(x^2 + y^2)}{x^2 + y^2} = |y|.
\]
So, using the triangle inequality,
\[
|f(x, y)| = \left| \frac{x^3}{x^2 + y^2} + \frac{y^3}{x^2 + y^2} \right| \leq \left| \frac{x^3}{x^2 + y^2} \right| + \left| \frac{y^3}{x^2 + y^2} \right| \leq |x| + |y|.
\]
We’re now ready to identify the functions \( g(x, y) \) and \( h(x, y) \). Let’s set
\[
g(x, y) = -|x| - |y| \quad \text{and} \quad h(x, y) = |x| + |y|.
\]
Then
\[
g(x, y) \leq f(x, y) \leq h(x, y)
\]
and
\[
\lim_{(x,y) \to (0,0)} g(x, y) = 0 = \lim_{(x,y) \to (0,0)} h(x, y),
\]
so \( \lim_{(x,y) \to (0,0)} f(x, y) = 0 \). Here’s a plot of \( z = f(x, y) \):
As before, there’s a hole at $(0,0)$, since $f(0,0)$ is not defined, but this singularity is removable\textsuperscript{1}, because $\lim_{(x,y)\to(0,0)} f(x,y)$ exists.

**Example 3.** (§15.2, Exercise 43 of the textbook) Suppose the following figure is a contour plot for $f(x,y)$:

![Contour Plot]

Explain why $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist.

**(Solution)** Each of the circles (or segments of circles) in the contour plot represents a different level curve for $f(x,y)$, meaning that $f(x,y)$ is constant along each of these circles, and we assume that the values assumed on those circles are distinct. That is, $f$ assumes a different value on the smallest circle than it does on the second-smallest circle. But this gives us two (in fact many) paths along which we may approach the point $(0,0)$ and obtain different limits. In particular, let $r_1(t)$ parametrize the smallest circle on the right side of the $y$-axis, and let $r_2(t)$ parametrize the second-smallest circle, with $r_1(0) = r_2(0) = (0,0)$. Then $f(r_1(t)) = c_1$ and $f(r_2(t)) = c_2$ are constant and distinct, so

$$c_1 = \lim_{t \to 0} f(r_1(t)) \neq \lim_{t \to 0} f(r_2(t)) = c_2.$$

So letting $(x,y)$ approach $(0,0)$ along different paths gives different values for the limit of $f(x,y)$, meaning that the limit does not exist.

**Example 4.** (§15.2, Exercise 48 of the textbook) Prove that if $f(x)$ is continuous at $x = a$ and $g(y)$ is continuous at $y = b$, then $F(x,y) = f(x)\, g(y)$ is continuous at $(a,b)$.

**(Solution)** Let’s remember the definition our book uses for continuity: we say that a function $F$ of two variables is continuous at $(a,b)$ if

$$\lim_{(x,y)\to(a,b)} F(x,y) = F(a,b). \quad (1)$$

The problem statement tells us that

$$\lim_{x \to a} f(x) = f(a) \quad \text{and} \quad \lim_{y \to b} g(y) = g(b).$$

\textsuperscript{1}This phrase basically means that we can usually just pretend that $f(x,y) = 0$ (since this is the limit we found) without getting into too much trouble.
Since \( F(x, y) = f(x)g(y) \), verifying (1) is really just a matter of applying the so-called ‘product law’ for limits. Let’s define \( f(x, y) := f(x) \) and \( g(x, y) := g(y) \). Then
\[
\lim_{(x,y) \to (a,b)} F(x, y) = \lim_{(x,y) \to (a,b)} f(x)g(y) = \lim_{(x,y) \to (a,b)} \bar{f}(x, y)\bar{g}(x, y)
\]
\[
= \left( \lim_{(x,y) \to (a,b)} \bar{f}(x, y) \right) \left( \lim_{(x,y) \to (a,b)} \bar{g}(x, y) \right)
\]
\[
= \left( \lim_{x \to a} f(x) \right) \left( \lim_{y \to b} g(y) \right)
\]
\[
= f(a)g(b) = F(a, b).
\]

The fourth equality (where our two-variable limits become single-variable limits) holds because \( \bar{f}(x, y) \) is independent of \( y \) and \( \bar{g}(x, y) \) is independent of \( x \). Because we’ve verified (1), we see that \( F \) is continuous at \((a, b)\). 

\[\Box\]

**Example 5.** (§15.2, Exercise 49 of the textbook) Let
\[
f(x, y) = \frac{x^3y}{x^6 + 2y^2}.
\]

Show that as \((x, y)\) approaches \((0, 0)\) along any line of the form \( y = mx \), \( f(x, y) \) limits to 0. Show that, in spite of this, \( \lim_{(x,y) \to (0,0)} f(x, y) \) does not exist.

*(Solution)* If \((x, y)\) is on the line \( y = mx \) then we write our point as \((x, mx)\), and letting \((x, y)\) approach \((0,0)\) amounts to letting \( x \) approach 0. So we’re interested in \( \lim_{x \to 0} f(x, mx) \). We have
\[
\lim_{x \to 0} f(x, mx) = \lim_{x \to 0} \frac{x^3(mx)}{x^6 + 2(mx)^2} = \lim_{x \to 0} \frac{mx^4}{x^6 + m^2x^2} = \lim_{x \to 0} \frac{mx^2}{x^4 + m^2} = 0.
\]

The last equality follows because the expression is continuous and we may simply evaluate at \( x = 0 \). So \( f(x, y) \) approaches 0 as \((x, y)\) approaches \((0,0)\) along any line of the form \( y = mx \). Approaching along the line \( x = 0 \) also sends \( f(x, y) \) to 0, so it’s tempting to conclude that the limit exists and is 0. But here’s a graph of the function, along with various paths of approach:
The blue paths represent approaches along the lines $y = mx$, and the green path depicts the curve $(x, x^3, f(x, x^3))$. That is, we let $(x, y)$ approach $(0, 0)$ along $y = x^3$. Let’s evaluate what happens here:

$$\lim_{x \to 0} f(x, x^3) = \lim_{x \to 0} \frac{x^3(x^3)}{x^6 + 2(x^3)^2} = \lim_{x \to 0} \frac{x^6}{3x^6} = \frac{1}{3}.$$ 

So approaching $(0, 0)$ along this path leads to a different limit, meaning that $\lim_{(x,y) \to (0,0)} f(x, y)$ does not exist.